Frequency characteristics of the fractional oscillator Van der Pol

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Abstract. Into this paper, the amplitude-frequency and phase-frequency characteristics of the Van der Polar fractional oscillator are studied in order to establish their relationship with the orders of fractional derivatives included in the model equation. Using the harmonic balance method, analytical formulas were obtained for the amplitude-frequency, phase-frequency characteristics, as well as the quality factor – the energy characteristic of the oscillatory system. It was shown that the quality factor depends on the orders of fractional derivatives, and change in their values can lead to both an increase and a decrease in the quality factor.

1 Introduction

At present, fractional oscillators, mathematical models of oscillatory systems with memory effects, are of great interest to researchers [1, 2]. This interest is caused by their application in various fields of knowledge, for example, in mechanics in the problem of beam oscillation in a fractal medium [3], in tribology – the lateral movement of the load along the fractal surface in an atomic force microscope or the movement of lithospheric plates relative to each other [4], in biophysics – the propagation of a nerve impulse in a memory membrane [5].

Fractional oscillators are described using fractional derivatives, which are studied in the framework of the theory of fractional integro-differentiation [6]. It was shown in [7, 8] that the orders of fractional derivatives are related to the quality factor of the oscillatory system and this is of very important practical importance. On the other hand, it was shown that fractional derivatives describe dissipative media well. In this paper, we investigated the forced oscillations of a fractional oscillator of Van der Pol – an analog of the Van der Pol oscillator. Analytical formulas are obtained for the amplitude-frequency (AFC) and phase-frequency (PFC) characteristics, as well as the quality factor of the oscillatory system.

2 Fractional Oscillator Van der Pol

Consider the fractional oscillator of Van der Pol:

$$\partial_{0t}^{\beta} x(\tau) + \lambda \left(x^2(t) - 1 \right) \partial_{0t}^{\gamma} x(\tau) + \omega_0^{\beta} x(t) = f \cos(\omega t), \; x(0) = x_0, \dot{x}(0) = y_0. \quad (1)$$

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where $x(t)$ - offset function, λ – nonlinear friction coefficient, ω_0^β – natural frequency, f and ω – amplitude and frequency of external harmonic influence, $t \in [0, T]$ – process time, $T > 0$ – process time, x_0 and y_0 – given constants, responsible for the initial condition, fractional derivatives in equation (1) are understood in the sense of Gerasimov-Caputo [9, 10]:

$$\partial_{0t}^{\beta} x(\tau) = \frac{1}{\Gamma(2-\beta)} \int_0^t \frac{\ddot{x}(\tau)\, d\tau}{(t-\tau)^{\beta-1}}, \partial_{0t}^{\gamma} x(\tau) = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{\dot{x}(\tau)\, d\tau}{(t-\tau)^{\gamma}}, 1<\beta<2, 0<\gamma<1. \quad (2)$$

Note, that in the particular case, where $\beta = 2$ and $\gamma = 1$ fractional oscillator goes into van der Paul oscillator:

$$\ddot{x}(t) + \lambda \left(x^2(t) - 1\right) \dot{x}(t) + \omega_0^\beta x(t) = f \cos(\omega t), x(0) = x_0, \dot{x}(0) = y_0.$$

Consider the characteristics of the Van der Pol fractional oscillator (1).

3 Characteristics of the Van der Paul fractional oscillator

In the future, the object of our study will be the steady-state oscillations of the fractional oscillator (1). Using the harmonic balance method, we derive analytical formulas for calculating the amplitude and phase of steady-state oscillations, as well as the quality factor of a fractional oscillator (1). We seek the solution of equation (1) according to the harmonic balance method in the form:

$$x(t) = A\cos(\omega t + \delta) = A\cos(u). \quad (3)$$

Taking into account (2) and (3), we rewrite equation (1) in the form:

$$\frac{1}{\Gamma(2-\beta)} \int_0^t \frac{\ddot{x}(\tau)\, d\tau}{(t-\tau)^{\beta-1}} + \frac{\lambda}{\Gamma(1-\gamma)} \int_0^t \frac{\dot{x}(\tau)\, d\tau}{(t-\tau)^{\gamma}} + \omega_0^\beta x(t) = f\cos(\omega t). \quad (4)$$

Then the first term in (4), taking into account (3), can be represented as:

$$\frac{1}{\Gamma(2-\beta)} \int_0^t \frac{\ddot{x}(\tau)\, d\tau}{(t-\tau)^{\beta-1}} = \frac{1}{\Gamma(2-\beta)} \int_0^t v^{1-\beta} \ddot{x}(t-v)\, dv = -\frac{A\omega^2}{\Gamma(2-\beta)} \int_0^t v^{1-\beta} \cos(\omega(t-v)+\delta) dv =$$

$$= -\frac{A\omega^2}{\Gamma(2-\beta)} \int_0^t v^{1-\beta} [\cos(\omega(t-v))\cos(v) - \sin(\omega(t-v))\sin(v)] dv =$$

$$= -\frac{A\omega^2 \cos(v)}{\Gamma(2-\beta)} \int_0^t v^{1-\beta} \cos(\omega(t-v))\, dv + \frac{A\omega^2 \sin(v)}{\Gamma(2-\beta)} \int_0^t v^{1-\beta} \sin(\omega(t-v))\, dv =$$

$$= -\frac{A\omega^2 \cos(v)}{\Gamma(2-\beta)} \int_0^t v^{1-\beta} [\cos(\omega t)\cos(\omega v) + \sin(\omega t)\sin(\omega v)]\, dv +$$

$$+ \frac{A\omega^2 \sin(v)}{\Gamma(2-\beta)} \int_0^t v^{1-\beta} [\sin(\omega t)\cos(\omega v) - \cos(\omega t)\sin(\omega v)]\, dv =$$

$$= -\frac{A\omega^2 \cos(v)\cos(\omega t)}{\Gamma(2-\beta)} \int_0^t v^{1-\beta} \cos(\omega v)\, dv - \frac{A\omega^2 \cos(v)\sin(\omega t)}{\Gamma(2-\beta)} \int_0^t v^{1-\beta} \sin(\omega v)\, dv +$$

$$+ \frac{A\omega^2 \sin(v)\sin(\omega t)}{\Gamma(2-\beta)} \int_0^t v^{1-\beta} \cos(\omega v)\, dv - \frac{A\omega^2 \sin(v)\cos(\omega t)}{\Gamma(2-\beta)} \int_0^t v^{1-\beta} \sin(\omega v)\, dv =$$

$$= [\sin(v)\sin(\omega t) - \cos(v)\cos(\omega t)] \frac{A\omega^2}{\Gamma(2-\beta)} \int_0^t v^{1-\beta} \cos(\omega v)\, dv -$$

$$- [\sin(v)\cos(\omega t) + \cos(v)\sin(\omega t)] \frac{A\omega^2}{\Gamma(2-\beta)} \int_0^t v^{1-\beta} \sin(\omega v)\, dv =$$

$$= \frac{-A\omega^2 \cos(u)}{\Gamma(2-\beta)} \int_0^t v^{1-\beta} \cos(\omega v)\, dv - \frac{A\omega^2 \sin(u)}{\Gamma(2-\beta)} \int_0^t v^{1-\beta} \sin(\omega v)\, dv. \quad (5)$$

In the case of steady-state oscillations at $t \to \infty$, integrals in (5) can be written as follows [6]:

$$\int_0^t v^{1-\beta} \cos(\omega v)\, dv \approx \frac{\Gamma(2-\beta)}{\omega^{2-\beta}} \sin((\beta-1)\pi/2) = -\frac{\Gamma(2-\beta)}{\omega^{2-\beta}} \cos(\beta\pi/2),$$

$$\int_0^t v^{1-\beta} \sin(\omega v)\, dv \approx \frac{\Gamma(2-\beta)}{\omega^{2-\beta}} \cos((\beta-1)\pi/2) = \frac{\Gamma(2-\beta)}{\omega^{2-\beta}} \sin(\beta\pi/2).$$

Given these relationships, we obtain:

$$\frac{1}{\Gamma(2-\beta)} \int_0^t \frac{\ddot{x}(\tau)\, d\tau}{(t-\tau)^{\beta-1}} \approx A\omega^\beta (\cos(u)\cos(\beta\pi/2) - \sin(u)\sin(\beta\pi/2)). \quad (6)$$

Similarly, the second term in (4) will have the form:

$$\frac{\lambda\left(x^2(t)-1\right)}{\Gamma(1-\gamma)} \int_0^t \frac{\dot{x}(\tau)\, d\tau}{(t-\tau)^\gamma} \approx \lambda A\omega^\gamma \left(\left(\frac{3}{4}A^2 - 1\right)\cos(\gamma\pi/2)\cos(u) - \left(\frac{1}{4}A^2 - 1\right)\sin(\gamma\pi/2)\sin(u)\right). \quad (7)$$

Taking into account (3), (6) and (7), we rewrite equation (3) in the form:

$$A\omega^\beta (\cos(u)\cos(\beta\pi/2) - \sin(u)\sin(\beta\pi/2)) +$$
$$+\lambda A\omega^\gamma \left(\left(\frac{3}{4}A^2 - 1\right)\cos(\gamma\pi/2)\cos(u) - \left(\frac{1}{4}A^2 - 1\right)\sin(\gamma\pi/2)\sin(u)\right) + \quad (8)$$
$$+\omega_0^\beta A\cos(u) = f\cos(\omega t).$$

We take into account that $\dot{x}(t) = -A\omega\cos(u)$, $\ddot{x}(t) = -A\omega^2\sin(u)$. herefore, equation (8) can be represented in the following form:

$$m\ddot{x}(t) + p\dot{x}(t) + s^2 x(t) = f\cos(\omega t), \quad (9)$$

$m = -\omega^{\beta-2}\cos(\beta\pi/2)$, $p = \omega^{\beta-1}\sin(\beta\pi/2) + \lambda\omega^{\gamma-1}\left(\frac{1}{4}A^2 - 1\right)\sin(\gamma\pi/2)$, $s^2 = \omega_0^\beta + \lambda\omega^\gamma\left(\frac{3}{4}A^2 - 1\right)\cos(\gamma\pi/2)$.

Note that equation (9) is a classical linear oscillator with friction and external harmonic influence, for which the relations for the amplitude-frequency (AFC) and phase-frequency (PFC) characteristics, as well as the quality factor, are known Q:

$$A = \frac{f}{\sqrt{U^2 + W^2}}, \delta = \arctan\left(-\frac{W}{U}\right), Q = \frac{s}{p}, \tag{10}$$

where $U = s^2 - \omega^2 m, W = \omega p$.

Note, all the characteristics in formulas (10) depend on the amplitude A of the steady-state oscillations. Consider the visualization of simulation results using formulas (10) using a specific example.

4 Results of the study

Let us the values of the parameters of the Van der Pol fractional oscillator for formulas (10) be chosen as follows: $f = 0.5$, $\lambda = 0.15$ and $\omega_0 = 1.5$. In fig. 1 shows the calculated frequency response curves for a fixed value $\gamma = 1$ and various values β.

It can be seen from Fig. 1 that when the fractional parameter $\beta \to 1$ changes, the amplitude of the forced oscillations decreases and the width of the resonance curve increases. Due to the fact that the quality factor Q of the oscillatory system is inversely proportional to the width of the resonance curve, we can conclude that the quality factor will decrease. This conclusion is confirmed by the surface in Fig. 2 constructed by formulas (10).

In fig. 3 shows the surface plotted by formulas (10) depending on a fixed parameter $\beta = 1.8$ for various values of the fractional parameter γ.

From fig. 3. it is seen that when the parameter $\gamma \to 1$, the quality factor increases. Therefore, we can conclude that the parameters β and γ affect the energy characteristic of the oscillatory system – quality factor.

In fig. 4. The surface characterizing the phase-frequency characteristic δ is shown – phase shift between the external action and steady-state oscillations depending on the frequency. When $\omega \to \infty$, in the classic case, $\delta \to -\pi$. In our case, the phase response also depends on the amplitude A, so when $A \to 0$ we also get in the limit $\delta \to -\pi$. If $A \to 3$, then the phase response $\delta \to 0$.

In fig. 5. With $A \to 1$ and $\omega \to 3$ $\delta \to 0$. Therefore, we can say that when the parameter changes β, the phase shift δ decreases.

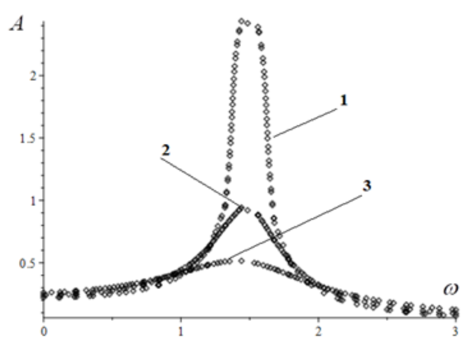

Figure 1. Resonance curves (AFC) plotted for various values β: $1 - \beta = 2$; $2 - \beta = 1.8$; 3 - $\beta = 1.6$

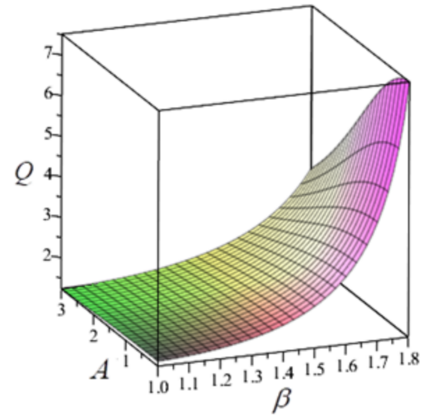

Figure 2. The dependence of the quality factor Q on the amplitude A of the forced oscillations and fractional order β

5 Acknowledgment

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6 Conclusion

Into the paper, using the harmonic balance method, analytical formulas were obtained for calculating the amplitude-frequency and phase-frequency characteristics, as well as the quality factor. It is shown that the orders of fractional derivatives affect the quality factor. It is confirmed that fractional derivatives describe dissipative media well

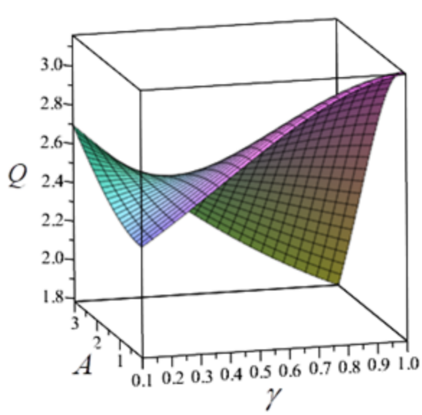

Figure 3. The dependence of the quality factor Q on the amplitude A of the forced oscillations and fractional order γ

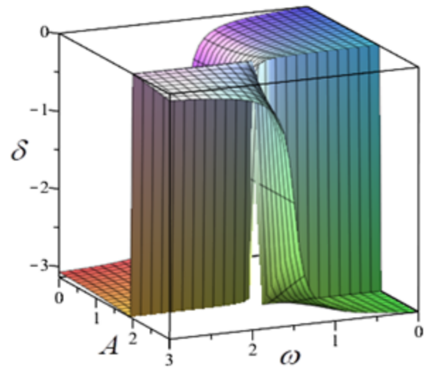

Figure 4. Phase-frequency characteristic δ depending on the amplitude A of the forced oscillations and frequency ω at $\beta = 2, \gamma = 1$

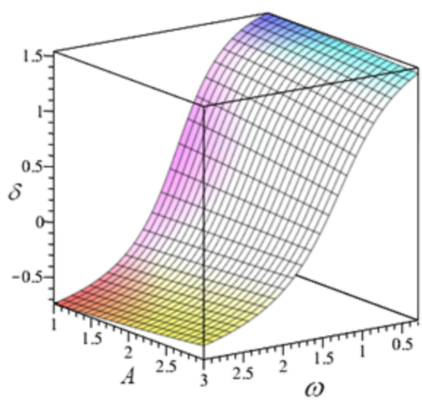

Figure 5. Phase-frequency characteristic δ depending on the amplitude A of the forced oscillations ω and frequency at $\beta = 1.6, \gamma = 1$

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