Charge Density Distribution Model in Self-Organizing Cloud

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Abstract. The paper considers a charge distribution model in a thundercloud presenting a self-organizing system in view of its fractal structure. An analytical solution to the model equation is obtained. Using numerical calculations, the distribution of charges in the fractal medium is shown and a comparative analysis of the existing models is carried out.

1 Introduction

To date, on the one hand, great successes have been achieved in solving the mysteries of atmospheric electricity [1-5], but on the other hand we still do not have a reliable picture of the thunderstorm electrification processes, they are still not fully understood. The modern knowledge of thunderstorms consists of the idea that cumulonimbus clouds extend at an altitude of 15 km, and the base cloud at an altitude of 0.3–3.5 km resulting in huge cumulonimbus "hot towers". When the air flow rises upward, the water vapor contained in it condenses to form water droplets and the heat energy is released. The water droplets then freeze while the surrounding air heats up. A huge mass of water and pieces of ice are retained in a thundercloud by updrafts with the velocities range 5-30 m/s. The updrafts entrain the warm air from the Earth's surface producing the thermal energy which is partly converted into electrical one.

Thus, the phase of lightning is preceded by the electrification of water droplets and ice particles, charges separation and accumulation in a thundercloud, which have logical impact on the total charge density [6, 7]. Therefore, the study of changes in charge density is very important from both the fundamental and practical points of view. One such important item is the mechanisms of space charges distribution within the thunderclouds, since the electric field depends on the charge density.

Considering the electric charge distribution, the fractal structure of the cloud should be taken into account [8-11]. In view of the fractal structure the use of the mathematical apparatus for integro-differentiation transforms the considered equation into a fractional differential equation.

This paper presents fractal modeling for the charge distribution within a self-organizing cloud (thundercloud) using the fractional integro-differentiation apparatus.

The electric charge density is considered, assuming that the process takes place in the clouds of monodisperse fine particles with the same charge considering the generation-recombination processes.

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2 Problem statement and solution

Consider the simplest electric field scheme in the monodisperse cloud assuming that it includes fine structure of cloud droplet concentration.
The formation of charged regions in thunderstorm clouds is preceded by processes leading to the electrification of cloud particles.

The accumulation of charges is possible if between certain regions of the thundercloud the electrification is quite intensive and prolonged while the charge recombination and dissipation is relatively slow. In this case, separation of charges can occur when charges on the particles having opposite signs move under gravity at different speeds [6].

There are many charge generation and separation theories that offer several operating mechanisms related to the cloud development phase and cloud in aggregated states. As mentioned above, the height of the base of thunderstorm clouds can reach the altitude of 0.3 - 3.5 km and the top of 15 km with several tens to hundreds of square meters in extent.

Objects such as a thundercloud are rightfully considered fractal, proceeding from the fact, that natural objects manifesting self-similarity in the range of at least two or three orders of spatial scales are fractals.

Let us study two electrification mechanisms specific to the thunderclouds regions located in the temperature zones below and above the Zero Isotherm. At a temperature above zero, large raindrops break up in the upward airflow. The electrical double layer in drop of water is formed at the liquid-air interface, on the droplet surface the charge is negative, which Ya. I. Frenkel explained by the fact that water is polar with the predominant orientation of molecules with negative ends outward. In the ascending moist air flow the water droplet loses its stability and breaks down into smaller and larger particles. As a result, smaller particles negatively charged rise with updrafts in the form of water dust, while the larger particles positively charged remain at the bottom cloud. At a temperature below zero the droplets begin to freeze accompanied by electrification. The surface of the droplet freezes first into a thin crust of ice. Due to the heat generated the temperature inside the droplet is maintained Zero. At a temperature difference between the drop center and its surface the ionic diffusion occurs in a way that the drop surface layer becomes positively charged by the hydrogen ions whose mobility nearly twice that of hydroxyl ions OH^-, and the drop core becomes negatively charged. Once the droplet center freezes the previously frozen surface layer bursts resulting from the droplet expansion, and positively charged fragments are released into the air. The residual particles left at the center of droplet or ice carrying a negative charge can later be crystallization centers, which collide with particles separated from droplets located above the zero isotherm and in a state of supercooling.

According to a large amount of research data, the charge on hail particles in a water aerosol supercooled stream is of the order of *1* electronic unit with the radius *R= 0.3 sm,* and the average charge generated by one droplet with radius *r = 30μm* is approximately $q_r = 4.4 \cdot 10^{-15}$ *Кл* , $q_r = 2 \cdot 10^{-15}$ *Кл* [12].

As mentioned above, water particles ripped off from the larger droplets carry the negative charge, but with respect to the air ionization while moving they can acquire the positive charge or simply entrain positive ions along. When these supercooled droplets collide with hailstones the crystallization front has a direction from the center to the edge. In this case, released gas small particles with the positive charge dissipate into the surrounding air [13], and the hailstone again acquires a negative charge.

Thus, we get a picture of a thundercloud with the lower layer containing positively charged drops, the center layer containing the negatively charged particles that are below the zero isotherm, and the upper layer containing the positively charged particles.

Let's select a column (with an ascending flow of charged particles), in a thundercloud of volume *V,* into which the charged particles pass through the region of the cloud base *S*. The full charge passing per unit of time through the given area, can be presented as

$$\int_s jdS\,, \tag{1}$$

where j - is the current density, dS - is the area element.

An electric current is also produced in the cloud column and corresponds to the initial amount of charge. Referring to the determination of the effective rate of change for some physical quantity f, introduced in view of the Riemann–Liouville fractional integral operator [14]

$$D_{at}^{\alpha}u(t)=\begin{cases}\dfrac{sign(t-a)}{\Gamma(-\alpha)}\displaystyle\int_a^t\frac{u(s)ds}{|t-s|^{\alpha+1}}, & \alpha<0,\\ u(t), & \alpha=0,\\ sign^n(t-a)\dfrac{\partial^n}{\partial t^n}D_{at}^{\alpha-n}u(t), & n-1<\alpha\le n, n\in N,\end{cases}$$

and also the Caputo derivative

$$\partial_{at}^{\alpha}u(t)=sign^n(a-t)D_{at}^{\alpha-n}\frac{\partial^n u(t)}{\partial t^n},\ n-1<\alpha\le n,\ n\in N,$$

where $\Gamma(z)$ – is Euler's gamma function, α –the fractional order (here $n=1$), get

$$\left\langle\frac{df}{dt}\right\rangle=\int_0^t g(t-t')\frac{df(t')}{dt'}dt'=\frac{1}{\tau}D_{0t}^{\alpha-1}\frac{df}{dt}=\frac{1}{\tau}\partial_{0t}^{\alpha}f,\quad 0<\alpha<1, \tag{2}$$

where $g(t)$ - memory function, t - is the dimensionless time or the characteristic time of the process τ, for current we can write down

$$I=-\frac{1}{\tau}\partial_{0t}^{\alpha}\int_V\rho dV\,, \tag{3}$$

where ρ - is the bulk charge density.

This type of velocity determination for a physical process with fractal structure in time was proposed and justified in [15]. Fractal property with respect to time is related to an external dissipative medium of a complex structure that include the investigated fractal thunderstorm clouds and the processes within. Equating (1) and (3), obtain

$$\int_S jdS=-\frac{1}{\tau}\partial_{0t}^{\alpha}\int_V\rho dV\,. \tag{4}$$

On the right-hand side (4), a permutation under the derivative sign is admissible since the function can be differentiated under the integral sign. Such method is well known as the Leibniz integral rule.

In view of the Ostrogradsky-Gauss theorem, write the following expression

$$\int_S jdS=\int_V divj\,dV\,. \tag{5}$$

Then by (4) and (5), obtain the continuity equation

$$\frac{1}{\tau}\partial_{0t}^{\alpha}\rho+divj=0\,. \tag{6}$$

In (6), the parameter α is a fractal dimension of the process with respect to the time variable. This implies ergodicity of the system. If the considered thundercloud is fractal, then the processes in it should be slower than in a similar continuous medium, thus $0<\alpha<1$.

The expression for the electric current density in the lower atmosphere can be written as

$$\lambda E+v\rho+D_T\nabla\rho+\sum_S j_S=j\,, \tag{7}$$

where ρ - is the electric charge density, λ - is the electrical conductivity, E - is the electric field strength, v - is the hydrodynamic flow velocity, D_T - is the turbulent diffusion coefficient, j_S - is the current density generated by the *i*-th source. The term $\sum_S j_S$ in (7) describes thunderstorms as an atmospheric current source. In our case $\sum_S j_S=0$.

Assuming that the fractal structure of a thunderstorm cloud is spatially homogeneous, and substituting (7) into (6), obtain

$$\frac{1}{\tau}\partial_{0t}^{\alpha}\rho+\lambda\nabla E+v\nabla\rho+D_T\Delta\rho=0,\ 0<\alpha<1\,. \tag{8}$$

In (8), relationship between atmospheric electrical conductivity λ and the ionic concentration $n_{1,2}^i$ is presented [3]

$$\lambda=\sum_i(e_i^1b_1^in_1^i+e_i^2b_2^in_2^i)\,,$$

where $b_{1,2}^i$ - is the the mobility of the *i*-th group ions, $n_{1,2}^i$ - is their volume concentration.

Considering the generation-recombination of the charge carriers equation (8) takes the form

$$\frac{1}{\tau}\partial_{0t}^{\alpha}\rho+\lambda\nabla E+v\nabla\rho+D_T\Delta\rho=e\left(G-R\right),\ 0<\alpha<1\,, \tag{9}$$

where e - is a charge, G and R are the functions for the generation and recombination processes.

Equation (9) is the main partial differential equation of fractional order proposed in this paper.

3 Results of calculations for the model testing

Let's rewrite equation (9) in the following form

$$\partial_{0t}^{\alpha}\rho-a\frac{\partial^2}{\partial x^2}\rho+b\frac{\partial}{\partial x}\rho=\tau e\left(\left(G-R\right)-\lambda\nabla E\right),\ 0<\alpha<1\,, \tag{10}$$

where $a=\tau D_T$, $b=\tau v$, $F=\tau e\left(\left(G-R\right)-\lambda\nabla E\right)$.

For equation (10), consider the Cauchy problem with an initial condition of the form $\rho\left(x,0\right)=\rho_0\left(x\right)$. To solve equation (10), make the following substitution

$$\rho\left(x,t\right)=\rho\left(\sqrt{a}\xi,t\right)=U\left(\xi,t\right), \tag{11}$$

where $x=\sqrt{a}\xi$, $U_{\xi}=\sqrt{a}\rho_x$, $U_{\xi\xi}=a\rho_{xx}$.

Then

$$\rho\left(x,0\right)=\rho\left(\sqrt{a}\xi,0\right)=\rho_0\left(\sqrt{a}\xi\right)=\rho_1\left(\xi\right). \tag{12}$$

$$F_1\left(\xi,t\right)=F\left(\sqrt{a}\xi,t\right)=F\left(x,t\right). \tag{13}$$

Substituting (11) in (10) and taking into account (12) and (13) we come to the equation

$$\partial_{0t}^{\alpha} U - \frac{\partial^2 U}{\partial \xi^2} + \frac{b}{\sqrt{a}} \frac{\partial U}{\partial \xi} = F_1(\xi, t),\ 0 < \alpha < 1. \tag{14}$$

Various boundary value problems for equations of the form (14) and methods for their solution are discussed in detail in the monograph [16]. Using the results of this work, we find the solution (14) in the form

$$U(\xi,t) = \int_{-\infty}^{\infty} \rho_1(\eta) D_{0t}^{\alpha-1} k(\xi-\eta,t)\,d\eta + \int_{-\infty}^{\infty}\int_0^t k(\xi-\eta,t-t_1) F_1(\eta,t_1)\,dt_1 d\eta, \tag{15}$$

where

$$k(\xi,t) = \frac{\exp\left(\frac{b\xi}{2\sqrt{a}}\right)}{2t} \int_{|\xi|}^{\infty} \phi\left(-\frac{\alpha}{2},0;-\frac{\sigma}{t^{\alpha/2}}\right) I_0\left(\frac{b\sqrt{\sigma^2-\xi^2}}{2\sqrt{a}}\right) d\sigma,$$

$$\phi(\alpha,\beta;t) = \sum_{n=0}^{\infty} \frac{t^n}{n!\Gamma(\alpha n+\beta)} \text{ - Wright function,}$$

$$I_0(t) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+1)} \left(\frac{t}{2}\right)^{2n} \text{ - Bessel function.}$$

Going to the initial parameters of our task

$$U(\xi,t) = U\left(\frac{x}{\sqrt{a}},t\right) = \rho(x,t).$$

Receive

$$\rho(x,t) = \int_{-\infty}^{\infty} \rho_1(\eta) D_{0t}^{\alpha-1} k\left(\frac{x}{\sqrt{a}}-\eta,t\right) d\eta + \int_{-\infty}^{\infty}\int_0^t k\left(\frac{x}{\sqrt{a}}-\eta,t-t_1\right) F_1(\eta,t_1)\,dt_1 d\eta \tag{16}$$

where

$$\rho_1(\eta) = \rho_0\left(\sqrt{a}\eta\right),\ F_1(\eta,t_1) = F_1\left(\sqrt{a}\eta,t_1\right),\ \sqrt{a}\eta = \eta_1,\ d\eta = \frac{d\eta_1}{\sqrt{a}},$$

$$I_1 = \int_{-\infty}^{\infty} \rho_0(\eta_1) D_{0t}^{\alpha-1} k\left(\frac{x-\eta_1}{\sqrt{a}},t\right)\frac{d\eta}{\sqrt{a}} = \int_{-\infty}^{\infty} \rho_0(\eta_1) D_{0t}^{\alpha-1} k_1(x-\eta_1,t)\,d\eta_1,$$

$$I_2 = \int_{-\infty}^{\infty}\int_0^t k\left(\frac{x-\eta_1}{\sqrt{a}},t-t_1\right) F(\eta_1,t_1)\frac{dt_1}{\sqrt{a}}\,d\eta_1 = \int_{-\infty}^{\infty}\int_0^t k_1(x-\eta_1,t-t_1) F(\eta_1,t_1)\,dt_1 d\eta_1,$$

$$\frac{1}{\sqrt{a}} k\left(\frac{x}{\sqrt{a}},t\right) = \frac{1}{\sqrt{a}} \frac{\exp\left(\frac{bx}{2a}\right)}{2t} \int_{\left|\frac{x}{\sqrt{a}}\right|}^{\infty} \phi\left(-\frac{\alpha}{2},0;-\frac{\sigma}{t^{\alpha/2}}\right) I_0\left(\frac{b\sqrt{\sigma^2-\frac{x^2}{a}}}{2\sqrt{a}}\right) d\sigma =$$

$$= \frac{1}{\sqrt{a}} \frac{\exp\left(\frac{bx}{2a}\right)}{2t} \int_{|x|}^{\infty} \phi\left(-\frac{\alpha}{2},0;-\frac{\sigma}{\sqrt{a}t^{\alpha/2}}\right) I_0\left(\frac{b}{2a}\sqrt{\sigma_1^2-x^2}\right)\frac{d\sigma_1}{\sqrt{a}}, \tag{17}$$

$$\sigma_1 = \sqrt{a}\sigma,\ \sigma = \frac{x}{\sqrt{a}},$$

$$\sigma_1 = x\,,\ d\sigma = \frac{d\sigma_1}{\sqrt{a}}\,.$$

As a result, the solution to our problem takes the following form

$$\rho(x,t) = \int_{-\infty}^{\infty} \rho_0(\eta_1) D_{0t}^{\alpha-1} k_1(x-\eta_1,t)\,d\eta_1 + \int_{-\infty}^{\infty}\int_0^t k_1(x-\eta_1,t-t_1)F(\eta_1,t_1)\,dt_1 d\eta_1\,, \quad (18)$$

where $k_1(x,t) = \frac{1}{a}\frac{\exp\left(\frac{bx}{2a}\right)}{2t}\int_{|x|}^{\infty}\phi\left(-\frac{\alpha}{2},0;-\frac{\sigma_1}{\sqrt{a}t^{\alpha/2}}\right)I_0\left(\frac{b}{2a}\sqrt{\sigma_1^2 - x^2}\right)d\sigma_1\,.$

To determine the degree of influence of the parameter on the charge density, consider a simplified version of the model for changing the charge density in a self-organizing fractal cloud environment, when the processes of generation and recombination are practically minimized, $F = 0$.

In a simplified version of the model, the solution can be presented as follows

$$\rho(x,t) = \frac{1}{2at^{\alpha}}\int_0^{\infty}\phi\left(-\frac{\alpha}{2},1-\alpha;-\frac{\sigma_1}{\sqrt{a}t^{\alpha/2}}\right)d\sigma_1 \times$$

$$\times\int_{-\sigma_1}^{\sigma_1}\rho_0(x-\eta_1)\exp\left(\frac{b\eta_1}{2a}\right)I_0\left(\frac{b}{2a}\sqrt{\sigma_1^2-\eta_1^2}\right)d\eta_1\,. \quad (19)$$

Figure 1 shows the results of numerical calculation using the formula (19) for $\alpha = 0.1$, $\alpha = 0.3$, $\alpha = 0.5$, $\alpha = 0.8$.

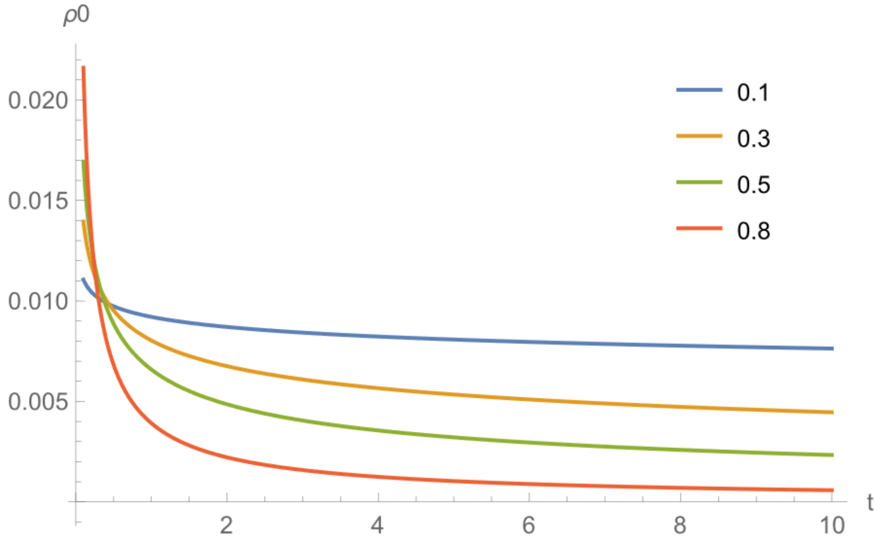

Fig.1. Charge density distribution in the cloud environment.

As the calculations show, a noticeable effect on the charge density occurs when the numerical value of the parameter decreases. This means that the fractal structure of thunderclouds should influence the density of charge distribution more strongly, the greater the fractality of the medium.

4 Conclusion

When studying electric charge distribution, fractal properties of the cloud medium should be considered together with various mechanisms involved in the formation of charged regions in thunderclouds preceded by processes leading to the electrification of clouds. Applying the mathematical apparatus of integro-differentiation, the equation under consideration transforms into a differential equation of fractional order.

The answers to the questions on electrical phenomena in thunderstorm clouds should be disclosed in terms of the cloud fractal structure. Electrical phenomena considered in this work, for the most part is still at the initial stage of study. The use of the apparatus of the theory of fractional integro-differential calculus contributed recently to fruitful researches. It helped to construct a charge distribution model within a self-organizing fractal cloud. The resulting model can be used to calculate the charge density of the cloud in view if its fractal structure. The numerical experiments have been carried out to evaluate the influence of the fractal structure on the charge distribution involving various combinations of microphysical parameters that showed the total charge density with respect to the fractal parameter.

In the following works, equation (9) will be considered, taking into account the right part, which already takes into account complex processes in storm clouds.

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