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Sufficient conditions for the uniqueness of the maxima of the optimization problem in the framework of a stochastic model with priorities depending on one random variable

N Neumerzhitskaia[1,*], *S* Uglich[1], and *T* Volosatova[1]

[1]Don State Technical University, 1, Gagarin square, Rostov-on-Don, 344000, Russia

Abstract. An objective function arising in the optimization problem of a quasilinear complex system with dependent priorities is considered. In the case of three priorities based on the results of one experiment, sufficient conditions are obtained for all stationary points of the objective function under consideration to be local maximum points.

1 Introduction

In papers [1–3] the problem of optimization of interaction within the framework of a unified system of a number of institutions and an “optimizer” interested in the successful functioning of the system and acting on the basis of expert assessments was considered. Expert assessments should be implemented in the choice of priorities (generally random and taking values between zero and one) with different distributions. Under certain natural conditions, a theorem on the existence and uniqueness of the local (and global) maximum of the objective function of a certain complex system consisting of functions of a quasilinear type and independent random priorities was proved. These maxima are functions of a finite number of the positive parameters. In the present work, the problem of minimizing these maxima in the domain of the parameters` definition is solved. In particular, an exact solution of the minimax problem for constant priorities is given. The theorem was proved that if the function of maximum arising in the optimization problem of a quasilinear complex system with independent priorities has a stationary point with respect to natural parameters, then this point is a local minimum point. This point is unique. It implements minimax of the objective function of the arbiter, whose task is to optimize the relationship of a number of the institutions with help of the expert assessments. In practice, expert judgments should enable a design of the priority’s distribution functions. And the arbiter sets up the mathematical model of the described complex system using the components of the obtained minimax point [4-10]. In this work, we began to investigate the described model, when the priorities are dependent random variables. This task turned out to be much more difficult. In the case of three priorities based on the results of one experiment, sufficient conditions are obtained for

* Corresponding author: neunata@yandex.ru

all stationary points of the objective function under consideration to be local maximum points..

2 Formulation of the problem

Let $(\Omega, \mathcal{F}, P)$ be a probability space. The mentioned problem of optimizing the relationship between the three institutions and the "arbiter" is based on the investigation of the maximum points of the objective function

$$F(u_1, u_2, u_3) = E\left(u_1^{\alpha_1} u_2^{\alpha_2} u_3^{\alpha_3}\right),\ u_1 > 0, u_2 > 0, u_3 > 0, \tag{1}$$

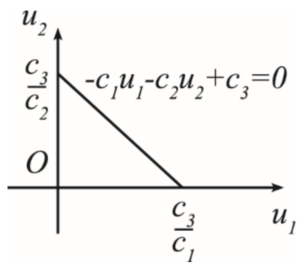

Fig. 1. All possible points of the local maximum of function (1) lie in the interior of the closed triangle.

where $\alpha_1, \alpha_2, \alpha_3$ are random variables (priorities) taking values on the interval $(0,1)$; E is the expectation with respect to the probability P; $u_3 = -c_1u_1 - c_2u_2 + c_3$, where c_1, c_2, c_3 are strictly positive parameters. Thus, all possible points of the local maximum of function (1) lie in the interior of the closed triangle shown in Figure 1.

It is clear that

$$D : \begin{cases} u_1 \geq 0 \\ u_2 \geq 0 \\ c_1u_1 + c_2u_2 \geq c_3 \end{cases}$$

and

$$D^0 : \begin{cases} u_1 > 0 \\ u_2 > 0 \\ c_1u_1 + c_2u_2 > c_3. \end{cases}$$

It is obvious that the function

$$F(u_1, u_2) := F(u_1, u_2, -c_1u_1 - c_2u_2 + c_3) \tag{2}$$

is continuous on D, continuously differentiable and strictly positive on D^0, and is equal to zero on the frontier ∂D of this domain. It follows from the well-known theorem of analysis that this function has at least one local maximum point, which is simultaneously a global maximum point. This work is devoted to the study of stationary points of function (2), provided that the priorities $\alpha_1, \alpha_2, \alpha_3$ are determined by one experiment associated with some uniformly distributed random variable α. So, we suppose that $\alpha_1 = f_1(\alpha)$, $\alpha_2 = f_2(\alpha)$, $\alpha_3 = f_3(\alpha)$, where f_1, f_2, f_3 are Borel functions $(0,1)$, taking values on the same interval $(0,1)$. By the change of variable theorem under the Lebesgue integral, we obtain:

$$F(u_1,u_2) = \int_0^1 u_1^{f_1(x)} u_2^{f_2(x)} (-c_1u_1 - c_2u_2 + c_3)^{f_3(x)} dx. \tag{3}$$

In this paper, a number of conditions on functions f_1, f_2, f_3 are obtained under which all stationary points of function (3) are points of local maximum.

3 Main results and their proofs

Introduce the notation:

$$\varphi(u_1,u_2,x) = u_1^{f_1(x)} u_2^{f_2(x)} (\text{-}c_1u_1 \text{ - } c_2u_2 \text{ - } c_3)^{f_3(x)}.$$

Then formula (3) takes the form:

$$F(u_1,u_2) = \int_0^1 \varphi(u_1,u_2,x)dx.$$

Introduce also the following notations:

$$\varphi_i(u_1,u_2) = \int_0^1 f_i(x)\varphi(u_1,u_2,x)dx,\ i=1,2,3,$$

$$\varphi_{ij}(u_1,u_2) = \int_0^1 f_i(x) f_j(x)\varphi(u_1,u_2,x)dx,\ i,j=1,2,3,\ i \le j.$$

Lemma 1. A point $(u_1,u_2) \in D^0$ is a stationary point of a function F if and only if the following equalities hold:

$$\begin{aligned} u_3 &= c_1u_1\frac{\varphi_3}{\varphi_1}, \\ u_3 &= c_2u_2\frac{\varphi_3}{\varphi_2}. \end{aligned} \tag{4}$$

Proof. The assertion of Lemma 1 follows immediately from the equalities:

$$\frac{\partial F}{\partial u_1} = \frac{1}{u_1}\varphi_1 - \frac{c_1}{u_3}\varphi_3,$$

$$\frac{\partial F}{\partial u_2} = \frac{1}{u_2}\varphi_2 - \frac{c_2}{u_3}\varphi_3.$$

Let us calculate the partial derivatives of the second order of the function F. We have:

$$\frac{\partial^2 F}{\partial u_1^2} = \frac{1}{u_1^2}(\varphi_{11} - \varphi_1) - \frac{2c_1}{u_1 u_3}\varphi_{13} + \frac{c_1^2}{u_3^2}(\varphi_{33} - \varphi_3),$$

$$\frac{\partial^2 F}{\partial u_2^2} = \frac{1}{u_2^2}(\varphi_{22} - \varphi_2) - \frac{2c_2}{u_2 u_3}\varphi_{23} + \frac{c_2^2}{u_3^2}(\varphi_{33} - \varphi_3), \tag{5}$$

$$\frac{\partial^2 F}{\partial u_1 \partial u_2} = \frac{1}{u_1 u_2}\varphi_{12} - \frac{c_1}{u_2 u_3}\varphi_{23} - \frac{c_2}{u_1 u_3}\varphi_{13} + \frac{c_1 c_2}{u_3^2}(\varphi_{33} - \varphi_3).$$

Let $(u_1, u_2) \in D^0$ be a stationary point of function F. Then equalities (4) hold. Substituting them in (5), we obtain:

$$\left(\frac{\partial^2 F}{\partial u_1^2}\right)_0 = -\frac{1}{u_1^2}(\varphi_1 - \varphi_{11}) - \frac{2}{u_1^2}\frac{\varphi_1 \varphi_{13}}{\varphi_3} - \frac{1}{u_1^2}\frac{\varphi_1^2}{\varphi_3^2}(\varphi_3 - \varphi_{33}),$$

$$\left(\frac{\partial^2 F}{\partial u_2^2}\right)_0 = -\frac{1}{u_2^2}(\varphi_2 - \varphi_{22}) - \frac{2}{u_2^2}\frac{\varphi_2 \varphi_{23}}{\varphi_3} - \frac{1}{u_2^2}\frac{\varphi_2^2}{\varphi_3^2}(\varphi_3 - \varphi_{33}),$$

$$\left(\frac{\partial^2 F}{\partial u_1 \partial u_2}\right)_0 = \frac{\varphi_{12}}{u_1 u_2} - \frac{1}{u_1 u_2}\frac{\varphi_1 \varphi_{23}}{\varphi_3} - \frac{1}{u_1 u_2}\frac{\varphi_2 \varphi_{13}}{\varphi_3} - \frac{1}{u_1 u_2}\frac{\varphi_1 \varphi_2}{\varphi_3^2}(\varphi_3 - \varphi_{33}).$$

Let us calculate at the stationary point the determinant

$$\Delta = u_1^2 u_2^2 \varphi_3^2 \begin{vmatrix} \left(\frac{\partial^2 F}{\partial u_1^2}\right)_0 & \left(\frac{\partial^2 F}{\partial u_1 \partial u_2}\right)_0 \\ \left(\frac{\partial^2 F}{\partial u_1 \partial u_2}\right)_0 & \left(\frac{\partial^2 F}{\partial u_2^2}\right)_0 \end{vmatrix}.$$

After a series of transformations, we get:

$$\begin{aligned} \Delta = {} & (\varphi_1 - \varphi_{11})(\varphi_2 - \varphi_{22})\varphi_3^2 + (\varphi_1 - \varphi_{11})(\varphi_3 - \varphi_{33})\varphi_2^2 + (\varphi_2 - \varphi_{22})(\varphi_3 - \varphi_{33})\varphi_1^2 \\ & + 2(\varphi_1 - \varphi_{11})\varphi_2\varphi_3\varphi_{23} + 2(\varphi_2 - \varphi_{22})\varphi_1\varphi_3\varphi_{13} + 2(\varphi_3 - \varphi_{33})\varphi_1\varphi_2\varphi_{12} \\ & + 2\varphi_1\varphi_2\varphi_{13}\varphi_{23} + 2\varphi_2\varphi_3\varphi_{12}\varphi_{13} + 2\varphi_1\varphi_3\varphi_{12}\varphi_{23} \\ & - \varphi_1^2\varphi_{23}^2 - \varphi_2^2\varphi_{13}^2 - \varphi_3^2\varphi_{12}^2. \end{aligned} \tag{6}$$

Lemma 2. A stationary point $(u_1, u_2) \in D^0$ of function F is a point of local maximum if in expression (6) $\Delta > 0$. Proof. Since for any i and j $\varphi_i > \varphi_{ij}$, it follows from (5) that at any point $(u_1, u_2) \in D^0$ the partial derivatives $\dfrac{\partial^2 F}{\partial u_1^2}$ and $\dfrac{\partial^2 F}{\partial u_2^2}$ are strictly negative. This implies the assertion of the lemma.

In all further statements, the fulfillment of the inequality $\Delta > 0$ will be verified, from which it follows that any stationary point of the function F is a local maximum point. In this case, it will be taken into account that in formula (6) all terms in the first three rows are strictly positive, and in the fourth row are strictly negative.

Proposition 1. Let us consider numbers a_i $\left(0 < a_i < 1,\ i = 1,2,3\right)$ and let $\varepsilon \le 0,05 \min a_i$. If $a_i - \varepsilon \le f_i \le a_i + \varepsilon$, then $\Delta > 0$.

Proof. We will ensure the inequality

$$2(a_1 - \varepsilon)(a_2 - \varepsilon)(a_3 - \varepsilon) \ge (a_1 + \varepsilon)(a_2 + \varepsilon)(a_3 + \varepsilon). \tag{7}$$

It is clear that for this it is sufficient to provide the inequalities

$$\frac{a_i - \varepsilon}{a_i + \varepsilon} \ge 0,9 \Leftrightarrow \varepsilon \le \frac{0,1 a_i}{1,9}, i = 1,2,3.$$

Obviously, this is true if we take $\varepsilon \le 0,05 \min a_i$. From (7) and the inequalities for f_i it follows that the system of inequalities

$$\begin{cases} \varphi_1\varphi_{23} + \varphi_2\varphi_{13} \ge \varphi_3\varphi_{12} \\ \varphi_2\varphi_{13} + \varphi_3\varphi_{12} \ge \varphi_1\varphi_{23} \\ \varphi_1\varphi_{23} + \varphi_3\varphi_{12} \ge \varphi_2\varphi_{13} \end{cases} \tag{8}$$

is satisfied and hence the system

$$\begin{cases} \varphi_3\varphi_{12}(\varphi_1\varphi_{23} + \varphi_2\varphi_{13}) \ge \varphi_3^2\varphi_{12}^2 \\ \varphi_1\varphi_{23}(\varphi_2\varphi_{13} + \varphi_3\varphi_{12}) \ge \varphi_1^2\varphi_{23}^2 \\ \varphi_2\varphi_{13}(\varphi_1\varphi_{23} + \varphi_3\varphi_{12}) \ge \varphi_2^2\varphi_{13}^2. \end{cases} \tag{9}$$

is satisfied too. It follows from (9) that

$$2\varphi_1\varphi_2\varphi_{13}\varphi_{23} + 2\varphi_2\varphi_3\varphi_{12}\varphi_{13} + 2\varphi_1\varphi_3\varphi_{12}\varphi_{23} \ge \varphi_1^2\varphi_{23}^2 + \varphi_2^2\varphi_{13}^2 + \varphi_3^2\varphi_{12}^2.$$

Hence the expression (6) is strictly positive and Proposition 3 is proved.

Corollary 1. If any $f_i, i = 1,2,3$ are constant, then $\Delta > 0$.

Proposition 2. 1) If two functions of $f_i, i = 1,2,3$ are constant, then $\Delta > 0$.

2) If one function of $f_i, i = 1,2,3,$ is arbitrary and the other two coincide and do not exceed $0,5$, then $\Delta > 0$.

Proof. Both points are proved by checking the fulfillment of inequalities (8) and further reasoning, as in Proposition 1.

Proposition 3. Let $f_1 \geq \frac{1}{2}, f_2 \geq \frac{1}{2}, f_3 \geq \frac{1}{2}$. Then $\Delta > 0.$

Proof. We have:

$$f_1 \geq \frac{1}{2} \Rightarrow 2\varphi_{13} \geq \varphi_3 \Rightarrow 2\varphi_2\varphi_{13} \geq \varphi_{12}\varphi_3 \Rightarrow 2\varphi_2\varphi_3\varphi_{12}\varphi_{13} \geq \varphi_3^2\varphi_{12}^2.$$

$$f_2 \geq \frac{1}{2} \Rightarrow 2\varphi_{12} \geq \varphi_1 \Rightarrow 2\varphi_{12}\varphi_3 \geq \varphi_1\varphi_{23} \Rightarrow 2\varphi_1\varphi_3\varphi_{12}\varphi_{23} \geq \varphi_1^2\varphi_{23}^2.$$

$$f_3 \geq \frac{1}{2} \Rightarrow 2\varphi_{23} \geq \varphi_2 \Rightarrow 2\varphi_1\varphi_{23} \geq \varphi_2\varphi_{13} \Rightarrow 2\varphi_1\varphi_2\varphi_{13}\varphi_{23} \geq \varphi_2^2\varphi_{13}^2.$$

The required result follows from (6).

Proposition 4. Let one of the functions $f_i, i = 1,2,3,$ be constant (for example, $f_3 = c$) and either $f_1 \geq \frac{c}{2(1+c)}$ or $f_2 \geq \frac{c}{2(1+c)}$. Then $\Delta > 0.$

Proof. It is easy to see that under the condition $f_3 = c$ the inequality

$$2\left(\varphi_3 - \varphi_{33}\right)\varphi_1\varphi_2\varphi_{12} + 2\varphi_1\varphi_2\varphi_{13}\varphi_{23} + 2\varphi_2\varphi_3\varphi_{12}\varphi_{13} + 2\varphi_1\varphi_3\varphi_{12}\varphi_{23} - \varphi_1^2\varphi_{23}^2 - \varphi_2^2\varphi_{13}^2 - \varphi_3^2\varphi_{12}^2 \geq 0$$

is equivalent to the inequality $\varphi_1\varphi_2 \geq \frac{c}{2(1+c)}\ \varphi_{12}F.$ The conclusion of this proposition follows from the last inequality and formula (6).

Corollary 2. Let one of the functions $f_i, i = 1,2,3$ be constant (for example, $f_3 = c$) and either $f_1 \geq \frac{1}{4}$ or $f_2 \geq \frac{1}{4}$. Then $\Delta > 0.$

Proof. The assertion follows from the inequality $\frac{1}{4} \geq \frac{c}{2(1+c)},$ which is valid for any $0 < c < 1.$

Proposition 5. Let $f_1 \leq \frac{1}{2}, f_2 \leq \frac{1}{2}, f_3 \leq \frac{1}{2}$. Then $\Delta > 0.$

Proof. Applying the inequalities in the formulation, we obtain

$$\varphi_1\varphi_2 = \varphi_1\frac{1}{2}\varphi_2 + \varphi_2\frac{1}{2}\varphi_1 \geq \varphi_1\varphi_{22} + \varphi_2\varphi_{11} \Rightarrow \varphi_1\varphi_2 - \varphi_1\varphi_{22} - \varphi_2\varphi_{11} \geq 0. \qquad (10)$$

By the Bunyakovsky-Schwartz inequality, we have

$$\varphi_{11}\varphi_{22} \geq \varphi_{12}^2. \qquad (11)$$

Adding (10) and (11), we obtain:

$$\varphi_1\varphi_2 - \varphi_1\varphi_{22} - \varphi_2\varphi_{11} + \varphi_{11}\varphi_{22} \geq \varphi_{12}^2 \Rightarrow (\varphi_1 - \varphi_{11})(\varphi_2 - \varphi_{22}) \geq \varphi_{12}^2 \Rightarrow$$
$$\Rightarrow (\varphi_1 - \varphi_{11})(\varphi_2 - \varphi_{22})\varphi_3^2 \geq \varphi_{12}^2\varphi_3^2.$$

The inequalities $(\varphi_2 - \varphi_{22})(\varphi_3 - \varphi_{33})\varphi_1^2 \geq \varphi_1^2\varphi_{23}^2$ and $(\varphi_1 - \varphi_{11})(\varphi_3 - \varphi_{33})\varphi_2^2 \geq \varphi_2^2\varphi_{13}^2$ are proved similarly. From (6) it follows that $\Delta > 0.$

Proposition 6. Let one of the functions $f_i, i = 1,2,3,$ be constant (for example, $f_3 = c$) and let there be a number $0 < \beta < 1$ such that $f_1 \leq \beta$ and $f_2 \leq 1 - \beta.$ Then $\Delta > 0.$

Proof. Since for $f_3 = c$ we have $2\varphi_1\varphi_2\varphi_{13}\varphi_{23} - \varphi_1^2\varphi_{23}^2 - \varphi_2^2\varphi_{13}^2 = 0$, then (as it follows from (6)) it suffices to prove the inequality $(\varphi_1 - \varphi_{11})(\varphi_2 - \varphi_{22}) - \varphi_{12}^2 \geq 0.$ We have:

$$\begin{cases} f_1^2 \leq \beta f_1 \\ f_2^2 \leq (1-\beta) f_2 \end{cases} \Rightarrow \begin{cases} \varphi_{11} \leq \beta\varphi_1 \\ \varphi_{22} \leq (1-\beta)\varphi_2 \end{cases} \Rightarrow \begin{cases} \varphi_2\varphi_{11} \leq \beta\varphi_1\varphi_2 \\ \varphi_1\varphi_{22} \leq (1-\beta)\varphi_1\varphi_2 \end{cases} \Rightarrow$$

$$\Rightarrow \varphi_2\varphi_{11} + \varphi_1\varphi_{22} \leq \varphi_1\varphi_2 \Rightarrow \varphi_1\varphi_2 - \varphi_2\varphi_{11} - \varphi_1\varphi_{22} \geq 0 \Rightarrow$$

$$\Rightarrow \varphi_1\varphi_2 - \varphi_2\varphi_{11} - \varphi_1\varphi_{22} + \varphi_{11}\varphi_{22} - \varphi_{12}^2 \geq 0 \Rightarrow$$

$$\Rightarrow (\varphi_1 - \varphi_{11})(\varphi_2 - \varphi_{22}) - \varphi_{12}^2 \geq 0.$$

Proposition 7. Let the inequalities $f_2 \geq f_3, 2f_3 \geq f_2;\ \ f_3 \geq f_1, 2f_1 \geq f_3;\ \ f_3 \geq 0{,}5$ be satisfied. Then $\Delta > 0.$

Proof.

$$1)\ \ f_2 \geq f_3,\ 2f_3 \geq f_2 \Rightarrow 2\varphi_2\varphi_{13} \geq \varphi_3\varphi_{12} \Rightarrow 2\varphi_2\varphi_3\varphi_{12}\varphi_{13} \geq \varphi_3^2\varphi_{12}^2.$$

$$2)\ \ f_3 \geq f_1, 2f_1 \geq f_3 \Rightarrow 2\varphi_3\varphi_{12} \geq \varphi_1\varphi_{23} \Rightarrow 2\varphi_1\varphi_3\varphi_{12}\varphi_{23} \geq \varphi_1^2\varphi_{23}^2.$$

$$3)\ \ f_3 \geq 0{,}5 \Rightarrow 2\varphi_{23} \geq \varphi_2 \Rightarrow 2\varphi_1\varphi_{23} \geq \varphi_2\varphi_{13} \Rightarrow 2\varphi_1\varphi_2\varphi_{13}\varphi_{23} \geq \varphi_2^2\varphi_{13}^2.$$

Applying formula (6) we obtain the required inequality.

4 Conclusion

The authors are confident that the inequality $\Delta > 0$ can be proved for arbitrary $f_i, i = 1,2,3,$ but so far this has not been possible. The next step in the investigation should be the solution of the minimax problem, as it was done in [2–3] with independent priorities.

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