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Research on some types of fractional differential equations which can be transformed into separable variables

Chiihuei Yu*

School of Mathematics and Statistics, Zhaoqing University, Guangdong, China

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Abstract. In this paper, we study some types of fractional differential equations which can be transformed into separable variables, regarding the Jumarie type of modified Riemann-Liouville fractional derivatives. We use a new multiplication of fractional functions and product rule for fractional derivatives to obtain the solutions of these fractional differential equations. Furthermore, some examples are given to demonstrate our results.

1 Introduction

Fractional calculus, a popular name used to denote the calculus of non-integer order, is as old as the calculus of integer order as created independently by Newton and Leibniz. In contrast with the calculus of integer order, fractional calculus has been granted a specific area of mathematics only in 1974, after the first international congress dedicated exclusively to it. Before this congress there were only sporadic independent papers, without a consolidated line [21-24]. It is nowadays well established that several real life phenomena are better described by fractional differential equations, where the term fractional, used for historical reasons, refers to derivative operators of any real positive order. Applications of fractional differential equations are commonly found in bioengineering, chemistry, control theory, electronic circuit theory, mechanics, physics, seismology, signal processing and so on ([1-20]). We refer to [8] for an historical perspective on fractional calculus. Unlike standard calculus, there is no unique definition of derivation and integration in fractional calculus. The commonly used definition is the Riemann-Liouville (R-L) fractional derivative [21]. Other useful definitions include Caputo definition of fractional derivative [22], the Grunwald-Letinikov (G-L) fractional derivative [23], and Jumarie's modified R-L fractional derivative [25].

In this paper, we study three types of fractional differential equations which can be transformed into separable variables, regarding the Jumarie type of modified R-L fractional derivatives. We define a new multiplication of fractional functions and use product rule for

* Corresponding author: 2019013010@zqu.edu.cn

fractional derivatives to solve these fractional differential equations. On the other hand, we give some examples to illustrate the methods used in this paper.

2 Preliminaries and methods

At first, the fractional calculus used in this article is introduced below.

Definition 2.1: Suppose that α is a real number and p is a positive integer. Then the modified Riemann-Liouville fractional derivatives of Jumarie type ([25]) is defined by

$$ {}_aD_x^{\alpha}[f(x)] = \begin{cases} \frac{1}{\Gamma(-\alpha)}\int_a^x (x-\tau)^{-\alpha-1} f(\tau)d\tau, & \text{if } \alpha < 0 \\ \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x (x-\tau)^{-\alpha}[f(\tau)-f(a)]\,d\tau & \text{if } 0 \le \alpha < 1 \\ \frac{d^p}{dx^p}\left({}_aD_x^{\alpha-p}\right)[f(x)], & \text{if } p \le \alpha < p+1 \end{cases} \quad (1)$$

where $\Gamma(\rho) = \int_0^{\infty} t^{\rho-1}e^{-t}dt$ is the gamma function defined on $\rho > 0$.

Proposition 2.2 ([26])**:** Let α, β, c be real numbers and $\beta \ge \alpha > 0$, then

$$ {}_0D_x^{\alpha}\left[x^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}, \quad (2)$$

and

$$ {}_0D_x^{\alpha}[c] = 0. \quad (3)$$

Definition 2.3 ([27])**:** The Mittag-Leffler function is defined by

$$ E_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)}, \quad (4)$$

where α is a real number, $\alpha > 0$, and z is a complex variable.

Next, a new multiplication of fractional functions is introduced below.

Definition 2.4 ([9])**:** Suppose that λ, μ, z are complex numbers, $0 < \alpha \le 1, j, l, k$ are non-negative integers, and a_k, b_k are real numbers, $p_k(z) = \frac{1}{\Gamma(k\alpha+1)}z^k$ for all k. The $\otimes$ multiplication is defined by

$$ p_j(\lambda x^{\alpha}) \otimes p_l(\mu y^{\alpha}) = \frac{1}{\Gamma(j\alpha+1)}(\lambda x^{\alpha})^j \otimes \frac{1}{\Gamma(l\alpha+1)}(\mu y^{\alpha})^l = \frac{1}{\Gamma((j+l)\alpha+1)}\binom{j+l}{j}(\lambda x^{\alpha})^j(\mu y^{\alpha})^l, \quad (5)$$

where $\binom{j+l}{j} = \frac{(j+l)!}{j!l!}$.

If $f(\lambda x^{\alpha})$ and $g(\mu y^{\alpha})$ are two fractional functions,

$$ f(\lambda x^{\alpha}) = \sum_{k=0}^{\infty} a_k\, p_k(\lambda x^{\alpha}) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(\lambda x^{\alpha})^k, \quad (6)$$

$$ g(\mu y^{\alpha}) = \sum_{k=0}^{\infty} b_k\, p_k(\mu y^{\alpha}) = \sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(\mu y^{\alpha})^k, \quad (7)$$

then we define

$$ f(\lambda x^{\alpha}) \otimes g(\mu y^{\alpha}) = \sum_{k=0}^{\infty} a_k\, p_k(\lambda x^{\alpha}) \otimes \sum_{k=0}^{\infty} b_k\, p_k(\mu y^{\alpha}) = \sum_{k=0}^{\infty}\left(\sum_{m=0}^{k} a_{k-m}b_m p_{k-m}(\lambda x^{\alpha}) \otimes p_m(\mu y^{\alpha})\right). \quad (8)$$

Proposition 2.5:

$$f(\lambda x^\alpha)\otimes g(\mu y^\alpha)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\,(\lambda x^\alpha)^{k-m}(\mu y^\alpha)^m. \quad (9)$$

Definition 2.6: Let $\left(f(\lambda x^\alpha)\right)^{\otimes n}=f(\lambda x^\alpha)\otimes\cdots\otimes f(\lambda x^\alpha)$ be the n times product of the fractional function $f(\lambda x^\alpha)$. If $f(\lambda x^\alpha)\otimes g(\lambda x^\alpha)=1$, then $g(\lambda x^\alpha)$ is called the $\otimes$ reciprocal of $f(\lambda x^\alpha)$, and denoted as $\left(f(\lambda x^\alpha)\right)^{\otimes -1}$.

Definition 2.7: If $\mathrm{f}(z)=\sum_{k=0}^{\infty}a_k z^k$, $g(\mu x^\alpha)=\sum_{k=0}^{\infty}b_k\,p_k(\mu x^\alpha)$, then

$$f_{\otimes\alpha}\left(g(\mu x^\alpha)\right)=\sum_{k=0}^{\infty}a_k\left(g(\mu x^\alpha)\right)^{\otimes k}. \quad (10)$$

The following is the major method used in this paper.

Theorem 2.8 (product rule for fractional derivatives) ([9])**:** If $0<\alpha\le 1$, λ,μ are complex numbers, and f , g are fractional functions. Then

$$\left({}_aD_x^\alpha\right)[f(\lambda x^\alpha)\otimes g(\mu x^\alpha)]=\left({}_aD_x^\alpha\right)[f(\lambda x^\alpha)]\otimes g(\mu x^\alpha)+f(\lambda x^\alpha)\otimes\left({}_aD_x^\alpha\right)[g(\mu x^\alpha)]. \quad (11)$$

3 Results and discussions

Here we mainly discuss three types of fractional differential equations which can be transformed into separable variables

Theorem 3.1: Let $0<\alpha\le 1$, then the first order homogeneous fractional differential equation

$${}_0D_x^\alpha[y]=g\left(y\otimes\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes -1}\right) \quad (12)$$

can be transformed into separable variables.

Proof Let $u=y\otimes\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes -1}$, then $y=u\otimes\frac{1}{\Gamma(\alpha+1)}x^\alpha$. Using product rule for fractional derivatives yields

$${}_0D_x^\alpha[y]=u+\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes\ {}_0D_x^\alpha[u]. \quad (13)$$

And hence,

$${}_0D_x^\alpha[u]=(g(u)-u)\otimes\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes -1}. \quad (14)$$

Since Eq. (14) is a separable variable fractional differential equation, the desired result holds.

Q.e.d.

Theorem 3.2: If $0<\alpha\le 1$, A,B,C are real numbers, then the first order fractional differential equation

$${}_0D_x^\alpha[y]=h\left(\frac{A}{\Gamma(\alpha+1)}x^\alpha+By+C\right) \quad (15)$$

can be transformed into separable variables.

Proof Let $u=\frac{A}{\Gamma(\alpha+1)}x^\alpha+By+C$, then

$${}_0D_x^\alpha[u]=A+B\cdot\ {}_0D_x^\alpha[y]. \quad (16)$$

Therefore,

$$ {}_0D_x^{\alpha}[u] = A + Bh(u). \tag{17} $$

Eq. (17) is a separable variable fractional differential equation, and hence the desired result holds.

Q.e.d.

Theorem 3.3: Assume that $0 < \alpha \leq 1$ and A, B, C, D, E, F are real numbers, then the first order fractional differential equation

$$ {}_0D_x^{\alpha}[y] = f\left(\left(\frac{A}{\Gamma(\alpha+1)}x^{\alpha} + By + C\right) \otimes \left(\frac{D}{\Gamma(\alpha+1)}x^{\alpha} + Ey + F\right)^{\otimes -1}\right). \tag{18} $$

can be transformed into separable variables.

Proof *Case 1.* If $C = F = 0$, then

$$ {}_0D_x^{\alpha}[y] = f\left(\left(\frac{A}{\Gamma(\alpha+1)}x^{\alpha} + By\right) \otimes \left(\frac{D}{\Gamma(\alpha+1)}x^{\alpha} + Ey\right)^{\otimes -1}\right) $$
$$ = f\left(\left(A + By \otimes \left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes -1}\right) \otimes \left(D + Ey \otimes \left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes -1}\right)^{\otimes -1}\right) $$
$$ = g\left(y \otimes \left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes -1}\right). \tag{19} $$

Thus, by Theorem 3.1, the desired result holds.

Case 2. If $C^2 + F^2 \neq 0$ and $\begin{vmatrix} A & B \\ D & E \end{vmatrix} \neq 0$. Then the system of equations

$$ \begin{cases} \frac{A}{\Gamma(\alpha+1)}x^{\alpha} + By + C = 0 \\ \frac{D}{\Gamma(\alpha+1)}x^{\alpha} + Ey + F = 0 \end{cases} \tag{20} $$

has a unique solution

$$ \begin{cases} \frac{1}{\Gamma(\alpha+1)}x^{\alpha} = \lambda \\ y = \mu \end{cases} \tag{21} $$

Let $\begin{cases} \frac{1}{\Gamma(\alpha+1)}X^{\alpha} = \frac{1}{\Gamma(\alpha+1)}x^{\alpha} - \lambda \\ Y = y - \mu \end{cases}$, then $\begin{cases} \frac{A}{\Gamma(\alpha+1)}X^{\alpha} + BY = 0 \\ \frac{D}{\Gamma(\alpha+1)}X^{\alpha} + EY = 0 \end{cases}$, and hence

$$ {}_0D_X^{\alpha}[Y] = f\left(\left(A + BY \otimes \left(\frac{1}{\Gamma(\alpha+1)}X^{\alpha}\right)^{\otimes -1}\right) \otimes \left(D + EY \otimes \left(\frac{1}{\Gamma(\alpha+1)}X^{\alpha}\right)^{\otimes -1}\right)^{\otimes -1}\right) $$
$$ = g\left(Y \otimes \left(\frac{1}{\Gamma(\alpha+1)}X^{\alpha}\right)^{\otimes -1}\right). \tag{22} $$

Therefore, by Theorem 3.1, the desired result holds.

Case 3. If $C^2 + F^2 \neq 0$ and $\begin{vmatrix} A & B \\ D & E \end{vmatrix} = 0$. There are three situations to discuss: (i) $A = B = 0$, then

Eq. (18) becomes ${}_0D_x^{\alpha}[y] = f\left(C \otimes \left(\frac{D}{\Gamma(\alpha+1)}x^{\alpha} + Ey + F\right)^{\otimes -1}\right)$; (ii) $D = E = 0$, then Eq. (18) becomes ${}_0D_x^{\alpha}[y] = f\left(\frac{1}{F} \otimes \left(\frac{A}{\Gamma(\alpha+1)}x^{\alpha} + By + C\right)\right)$; (iii) $\frac{A}{D} = \frac{B}{E} = k$, let $u =$

$\frac{D}{\Gamma(\alpha+1)}x^{\alpha}+Ey$, then ${}_0D_x^{\alpha}[u]=D+E\,{}_0D_x^{\alpha}[y]=D+Ef\left((ku+C)\otimes(u+F)^{\otimes-1}\right)=g(u)$ which is a separable variable fractional differential equation.

4 Examples

For the three types of first order fractional differential equations discussed in this article, we will give some examples and find their solutions.

Example 4.1: Consider the first order $1/3$- fractional differential equation

$$ {}_0D_x^{1/3}[y]=3y\otimes\left(\frac{1}{\Gamma(4/3)}x^{1/3}\right)^{\otimes-1}+2\left(y\otimes\left(\frac{1}{\Gamma(4/3)}x^{1/3}\right)^{\otimes-1}\right)^{\otimes2}. \quad (23)$$

Let $u=y\otimes\left(\frac{1}{\Gamma(4/3)}x^{1/3}\right)^{\otimes-1}$, then $y=u\otimes\frac{1}{\Gamma(4/3)}x^{1/3}$. By product rule for fractional derivatives, we have ${}_0D_x^{1/3}[y]=u+\frac{1}{\Gamma(4/3)}x^{1/3}\otimes\,{}_0D_x^{1/3}[u]$. Thus,

$$ {}_0D_x^{1/3}[u]=\left(\frac{1}{\Gamma(4/3)}x^{1/3}\right)^{\otimes-1}\otimes\left(2u+2u^{\otimes2}\right). \quad (24)$$

Hence, we obtain the general solution of Eq. (23)

$$ y=c\left(\frac{1}{\Gamma(4/3)}x^{1/3}\right)^{\otimes2}\otimes\left(\frac{1}{\Gamma(4/3)}x^{1/3}+y\right), \quad (25)$$

and the particular solution $y=-\frac{1}{\Gamma(4/3)}x^{1/3}$, where c is a constant.

Example 4.2: We study the first order $1/2$- fractional differential equation

$$ {}_0D_x^{1/2}[y]=\left(\frac{1}{\Gamma(3/2)}x^{1/2}+y\right)^{\otimes2}. \quad (26)$$

Let $v=\frac{1}{\Gamma(3/2)}x^{1/2}+y$, then ${}_0D_x^{1/2}[v]=1+\,{}_0D_x^{1/2}[y]$. So,

$$ {}_0D_x^{1/2}[v]=1+v^{\otimes2}. \quad (27)$$

And hence, the general solution of Eq. (26) is

$$ \frac{1}{\Gamma(3/2)}x^{1/2}+y=tan_{1/2}\left(\frac{1}{\Gamma(3/2)}x^{1/2}+c\right), \quad (28)$$

where c is a constant.

Example 4.3: Consider the first order $1/4$- fractional differential equation

$$ {}_0D_x^{1/4}[y]=\left(\frac{1}{\Gamma(5/4)}x^{1/4}-y+1\right)\otimes\left(\frac{1}{\Gamma(5/4)}x^{1/4}+y-3\right)^{\otimes-1}. \quad (29)$$

Let $\begin{cases}\frac{1}{\Gamma(5/4)}X^{1/4}=\frac{1}{\Gamma(5/4)}x^{1/4}-1\\ Y=y-2\end{cases}$, then we obtain

$$ {}_0D_X^{1/4}[Y]=\left(\frac{1}{\Gamma(5/4)}X^{1/4}-Y\right)\otimes\left(\frac{1}{\Gamma(5/4)}X^{1/4}+Y\right)^{\otimes-1}. \quad (30)$$

Let $u=Y\otimes\left(\frac{1}{\Gamma(5/4)}X^{1/4}\right)^{\otimes-1}$, then $Y=u\otimes\frac{1}{\Gamma(5/4)}X^{1/4}$. Thus,

$$ {}_0D_X^{1/4}[u]=\left(\frac{1}{\Gamma(5/4)}X^{1/4}\right)^{\otimes-1}\otimes\left(1-2u-u^{\otimes2}\right)\otimes(1+u)^{\otimes-1}. \quad (31)$$

Therefore, the general solution of Eq. (31) is

$$Y^{\otimes 2}+\frac{2}{\Gamma(5/4)}X^{1/4}\otimes Y-\left(\frac{1}{\Gamma(5/4)}X^{1/4}\right)^{\otimes 2}=c_1 \ . \tag{32}$$

Hence, we obtain the general solution of Eq. (29)

$$y^{\otimes 2}+\frac{2}{\Gamma(5/4)}x^{1/4}\otimes y-\left(\frac{1}{\Gamma(5/4)}x^{1/4}\right)^{\otimes 2}-6y-\frac{2}{\Gamma(5/4)}x^{1/4}=c \ , \tag{33}$$

where c is a constant.

5 Conclusions

As mentioned above, we can obtain the solutions of three types of first order fractional differential equations studied in this paper by using product rule for fractional derivatives. In fact, the application of product rule is extensive, and can be used to easily solve many fractional differential equations. On the other hand, our results are generalizations of classical first order differential equations which can be transformed into separable variables. In the future, we will use the Jumarie's modified R-L fractional derivatives and the new multiplication defined in this article to expand our research topics to the problems of applied mathematics and fractional calculus.

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