# Optimal control for a size-structured predator-prey model in a polluted environment

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**Abstract.** In this paper, we deal with an optimal harvesting problem for a periodic predator-prey hybrid system dependent on size-structure in a polluted environment. In other words, a size-dependent model in an environment with a small toxicant content has been established. The well-posedness of state system is proved by using the fixed point theorem. The necessary optimality conditions are derived by tangent-normal cone technique in nonlinear functional analysis. The existence of a unique optimal harvesting policy is verified via the Ekeland's variational principle. The optimal harvesting problem has an optimal harvesting problem. Using the optimization theories and methods in mathematics to control phenomena of life. The objective function represents the total economic profit from the harvested population. Some theoretical results obtained in this paper provide a scientific theoretical basis for the practical application of the model.

## **1** Introduction

With the rapid development of the global economy, many new technologies have been applied to industrial and agricultural production activities. The invasion of toxicants is very easy to happen, and environmental pollution has become more and more serious. Bush fires in Australia, SARS, Ebola virus, AIV, H1N1 influenza, and COVID-19, etc., these phenomena are threatening the ecological balance and causing serious harm to the survival of human beings and other creatures. Therefore, it is necessary to study the effects of toxicants on biological populations. Hallam et al. proposed an idea of using dynamics methods to study ecotoxicology in the literature [1-3]. They established a toxicants-population model and studied the persistence and extinction of a population in a polluted environment. Since the 1980s, people have conducted in-depth research on the topic of ecotoxicology. Nowadays, there is a large number of literature on ecotoxicology problems [4-6], but sizestructured factors are not considered in these models.

For many populations, the individual size determines the life parameters of the individual to a large extent, such as reproductive rate, mortality, metabolic capacity and predation capacity, etc., thereby affecting the dynamic behavior of the populations. In addition, the individual size-structure can explain some phenomena that other structures cannot explain, such as the selfthinning of plants and the elastic growth of individuals. Furthermore, the individual size has good operability, such as easy measurement, convenient scientific management of ecological resources, and easy access to size-based statistical data. It is more convenient in practical applications. This kind of models has achieved remarkable results through theory, numerical calculation and experimental methods, which can be found in the literature [7-10]. Due to the influence of seasonal changes and other factors, the living environment of the populations often experience periodic changes. For the research on the optimal harvest problems dependent on individual size-structured models in a periodic environment can be referred to [11-13], in which the literature [11] discussed harvesting problem for nonlinear size-dependent population model in periodic environments. At present, only a few papers have focused on the optimal control problems of population models with size-structure and periodic effects in a polluted environment. Inspired by the above, this paper discusses the optimal harvesting problem of a periodic predator-prey system dependent on size-structure in a polluted environment.

The remaining part of this paper is as follows. The problem is described and the main methods of proof are given in the next section. The well-posedness is proved, the optimality conditions for the harvesting problem are derived, the existence of a unique optimal policy is obtained and the problem is discussed in Sect. 3. Finally, we give a short conclusion in the last section.

## 2 Methodology

We establish the following periodic predator-prey model with size-structure in a polluted environment:

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$$\begin{cases} \frac{\partial p_{1}(s,t)}{\partial t} + \frac{\partial (g_{1}(s)p_{1}(s,t))}{\partial s} = f_{1}(s,t) - \mu_{1}(s,c_{10}(t)) \\ \cdot p_{1}(s,t) - \lambda_{1}(s,t)P_{2}(t)p_{1}(s,t) - u_{1}(s,t)p_{1}(s,t), \\ \frac{\partial p_{2}(s,t)}{\partial t} + \frac{\partial (g_{2}(s)p_{2}(s,t))}{\partial s} = f_{2}(s,t) - \mu_{2}(s,c_{20}(t)) \\ \cdot p_{2}(s,t) + \lambda_{2}(s,t)P_{1}(t)p_{2}(s,t) - u_{2}(s,t)p_{2}(s,t), \\ \frac{dc_{i0}(t)}{dt} = kc_{e}(t) - gc_{i0}(t) - mc_{i0}(t), \\ \frac{dc_{e}(t)}{dt} - k_{1}c_{e}[P_{1}(t) + P_{2}(t)] \\ + g_{1}[c_{10}(t)P_{1}(t) + c_{20}(t)P_{2}(t)] - hc_{e}(t) + v(t), \\ g_{i}(0)p_{i}(0,t) = \int_{0}^{t} \beta_{i}(s,c_{i0}(t))p_{i}(s,t)ds, \\ p_{i}(s,t) = p_{i}(s,t+T), \\ P_{i}(t) = \int_{0}^{t} p_{i}(s,t)ds, \quad i = 1,2, \end{cases}$$

where  $Q = (0, l) \times R_+, R_+ = (0, +\infty), l \in R_+$  is the maximal size,  $T \in R_+$  is the habitat evolution period. Other parameters are defined as follows:

 $p_i(s,t)$ : the density of ith population of size *s* at time *t*.

 $c_{i0}(t)$ : the concentration of the toxicants in ith population.

 $c_e(t)$ : the concentration of the toxicants in the environment.

The vital rates of ith populations  $g_i(s)$ ,  $\mu_i(s,t)$  and  $\beta_i(s,t)$ : the growth rate of the individual size, mortality and fertility, respectively.

 $\lambda_i(s,t)$ : the interaction coefficient.

v(t): the exogenous toxicants input rate.

 $P_i(t)$ : total number of ith population.

The k, g, m,  $k_1$ ,  $g_i$ , h are non-negative constants.

The  $u_i(s,t)$  is the harvesting efforts, which belongs to

$$\Omega = \left\{ (u, v) \in \left[ L_T^{\infty}(Q) \right]^2 \times L_T(R_+) : 0 \le u_i(s, t) \le H , \\ 0 \le v(t) \le h, \forall (s, t) \in Q \right\},$$

where  $\Omega$  is admissible control set, and  $\eta(s,t) = \eta(s,t+T)$ , a.e.  $(s,t) \in Q$ ,  $\eta(t) = \eta(t+T)$ , a.e.  $t \in (0,T)$ . In this paper, we consider the following optimal harvesting problem:

$$\max_{(u,v)\in\Omega} \left\{ J(u,v) : u = (u_1(s,t), u_2(s,t)), v = v(t) \right\}, \quad (2)$$

where

$$J(u,v) = \sum_{i=1}^{2} \int_{0}^{T} \int_{0}^{t} w_{i}(s,t) u_{i}(s,t) p_{i}(s,t) ds dt$$
$$-\frac{1}{2} \sum_{i=1}^{2} \int_{0}^{T} \int_{0}^{t} c_{i} \left[ u_{i}(s,t) \right]^{2} ds dt - \frac{1}{2} \int_{0}^{T} c_{3} \left[ v(t) \right]^{2} dt ,$$

represents the total economic profit from the harvested population. The weight function  $w_i(s,t)$  is the selling

price of an individual. The positive constant  $c_1$ ,  $c_2$  and  $c_3$  are the cost factors of harvesting populations and the cost factor of curbing environmental pollution, respectively. The state  $(p, c_0, c_e)$  is the solution of the system (1) corresponding to (u, v).

Throughout this paper, we assume that:

 $(A_1) g_i \in C^1[0,l), g_i(0) = 1, \lim_{s \uparrow l} g_i(s) = 0, g_i(s) > 0,$ and  $g_i(s) \le 0$ , a. e.  $s \in [0,l)$ . There are constants  $L_{V_i} > 0$  such that for all  $s_1, s_2 \in [0,l)$ :

$$\begin{aligned} \left| g_{i}(s_{1}) - g_{i}(s_{2}) \right| &\leq L_{V_{i}} \left| x_{1} - x_{2} \right|. \\ (A_{2}) &0 \leq \beta_{i}(s, c_{i0}(t)) \leq \beta^{0}(s, c_{i0}(t+T)), \ 0 \leq w_{i}(s, t) = \\ w_{i}(s, t+T) \leq \overline{w} \ . \\ (A_{3}) &\mu_{i}(\cdot, c_{i0}(t)) \in L_{loc}^{1}(0, l), 0 \leq \mu_{i}(s, c_{i0}(t)) = \mu_{i}(s, c_{i0}(t+T)), \\ \int_{0}^{l} \mu_{i}\left(\Gamma_{i}^{-1}\left(\Gamma_{i}(l) - s\right), t - s\right) ds = +\infty \ . \ \Gamma_{i}(s) = \int_{0}^{s} \frac{1}{g_{i}(\theta)} d\theta \ . \\ (A_{4}) &v(\cdot) \in L^{2}[0, T], \ 0 \leq v_{0} \leq v(t) \leq v_{1} < +\infty \ . \\ (A_{5}) &g \leq k \leq g + m, \ v_{1} \leq h \ . \\ (A_{6}) &\left|\beta_{i}(s, c_{i0}^{1}(t)) - \beta_{i}(s, c_{i0}^{2}(t))\right| \leq L_{\beta} \left|c_{i0}^{1}(t) - c_{i0}^{2}(t)\right|, \ \left|\mu(s, c_{i0}^{1}(t)) - \mu_{i}(s, c_{i0}^{2}(t))\right| \leq L_{\mu} \left|c_{i0}^{1}(t) - c_{i0}^{2}(t)\right|, \ (A_{7}) &f_{i} \in L^{\infty}(Q), \ 0 \leq f_{i} \leq B \ , \ f_{i}(s, t) = f_{i}(s, t+T) \ , \ \text{for} \\ (s, t) \in Q \ . \end{aligned}$$

Next, we will prove the existence and uniqueness of non-negative solution of the toxicant-population model by the fixed point theorem, deduct the optimality conditions of the optimal harvesting problem by employing tangent-normal cone techniques, and obtain the existence of optimal harvesting policy by the Ekeland's variational principle.

### 3 Experimental results and discussion

#### 3.1 Well-posedness of the state system

In this section, we discuss the well-posedness of state system (1), and first give the following definition.

**Definition 3.1.1** The  $(p_1(s,t), p_2(s,t), c_{10}(t), c_{20}(t), c_e(t))$  is a solution of state system (1), if it is absolutely continuous along almost every characteristic curve  $t - \Gamma_i(s) = \gamma$  ( $\gamma \in \mathbf{R}$ ) [14] and satisfies:

$$\begin{cases} \frac{\partial p_{1}(s,t)}{\partial t} + \frac{\partial (g_{1}(s)p_{1}(s,t))}{\partial s} = f_{1}(s,t) - \mu_{1}(s,c_{10}(t)) \\ \cdot p_{1}(s,t) - \lambda_{1}(s,t)P_{2}(t)p_{1}(s,t) - u_{1}(s,t)p_{1}(s,t), \\ \frac{\partial p_{2}(s,t)}{\partial t} + \frac{\partial (g_{2}(s)p_{2}(s,t))}{\partial s} = f_{2}(s,t) - \mu_{2}(s,c_{20}(t)) \\ \cdot p_{2}(s,t) + \lambda_{2}(s,t)P_{1}(t)p_{2}(s,t) - u_{2}(s,t)p_{2}(s,t), \\ \frac{dc_{i0}(t)}{dt} = kc_{e}(t) - gc_{i0}(t) - mc_{i0}(t), \\ \frac{dc_{e}(t)}{dt} - k_{1}c_{e}[P_{1}(t) + P_{2}(t)] \\ + g_{1}[c_{10}(t)P_{1}(t) + c_{20}(t)P_{2}(t)] - hc_{e}(t) + v(t), \\ \lim_{\gamma \to 0^{+}} g_{i}(\Gamma_{i}^{-1}(\gamma))p_{i}(\Gamma_{i}^{-1}(\gamma), t + \gamma) \\ = \int_{0}^{l} \beta_{i}(s,c_{i0}(t))p_{i}(s,t)ds, \\ p_{i}(s,t) = p_{i}(s,t+T), \\ P_{i}(t) = \int_{0}^{l} p_{i}(s,t)ds, \quad i = 1, 2, \end{cases}$$

**Theorem 3.1.2** Assuming  $(A_1)$ - $(A_7)$  hold, then the state system (1) has unique non-negative solution that satisfies the following conditions:

$$\begin{split} &(\mathrm{I}) \ \left( p_1(s,t), p_2(s,t), c_{10}(t), c_{20}(t), c_e(t) \right) \! \in \! \left[ L^{\infty}_T(Q) \right]^2 \! \times \! \left[ L^{\infty}(R_+) \right]^3; \\ &(\mathrm{II}) \ 0 \leq c_0(t) \leq 1, 0 \leq c_e(t) \leq 1, t \in R_+ \, . \end{split}$$

**Proof** Without a loss of generality, suppose  $u_i(x,t) \equiv 0$ . From definition 3.1.1, the solution of the state system (1) can be written as n(s,t)

$$p_{i}(s,t) = b_{i}(t - \Gamma_{i}(s)) \exp\left\{-\int_{0}^{\Gamma_{i}(s)} \mu_{i}\left(\Gamma_{i}^{-1}(\tau), c_{i0}(t + \tau - \Gamma_{i}(s))\right) + (-1)^{i+1} \lambda_{i}\left(\Gamma_{i}^{-1}(\tau), t + \tau - \Gamma_{i}(s)\right)P_{j}(t + \tau - \Gamma_{i}(s)) + g_{is}\left(\Gamma_{i}^{-1}(\tau)\right)\right\} d\tau + \int_{0}^{\Gamma_{i}(s)} \exp\left\{-\int_{\tau}^{\Gamma_{i}(s)} \mu_{i}\left(\Gamma_{i}^{-1}(\sigma), c_{i0}(t + \sigma - \Gamma_{i}(s))\right) + (-1)^{i+1} \lambda_{i}\left(\Gamma_{i}^{-1}(\sigma), t + \sigma - \Gamma_{i}(s)\right)P_{j}(t + \sigma - \Gamma_{i}(s)) + g_{is}\left(\Gamma_{i}^{-1}(\sigma)\right) d\sigma\right\} f_{i}\left(\Gamma_{i}^{-1}(\tau), t + \tau - \Gamma_{i}(s)\right) d\tau , i \neq j = 1, 2, \Gamma_{i}(s) \leq t ,$$
(3)

$$c_{i0}(t) = c_{i0}(0) \exp\{-(g+m)t\} + k \int_{0}^{t} c_{e}(s) \exp\{(s-t)(g+m)\} ds, \qquad (4)$$

$$c_{e}(t) = c_{e}(0) \exp\left\{-\int_{0}^{t} (k_{1}P_{1}(\tau) + k_{1}P_{2}(\tau) + h) d\tau\right\}$$
  
+ 
$$\int_{0}^{t} (g_{1}c_{10}(s)P_{1}(s) + g_{1}c_{20}(s)P_{2}(s) + v(s))$$
  
$$\cdot \exp\left\{\int_{t}^{s} (k_{1}P_{1}(\tau) + k_{1}P_{2}(\tau) + h) d\tau\right\} ds , \qquad (5)$$

where  $b_i(t) = p_i(0,t)$ , we obtain

$$\begin{split} b_{1}(t) &= g_{1}(0) p_{1}(0,t) \\ &= \int_{0}^{\Gamma_{1}^{-1}(t)} \beta_{1}(s,c_{10}(t)) p_{1}(s,t) ds \\ &\leq \int_{0}^{\Gamma_{1}^{-1}(t)} \beta_{1}(s,c_{10}(t)) \Big[ b_{1}(t-\Gamma_{1}(s)) \\ &+ \int_{0}^{\Gamma_{1}(s)} f_{1}(\Gamma_{1}^{-1}(\tau),c_{10}(t+\tau-\Gamma_{1}(s))) d\tau \Big] ds \\ &\leq \beta^{0} \int_{0}^{t} b_{1}(\sigma) d\sigma + \beta^{0} \|f_{1}(\cdot,\cdot)\|_{L^{1}(Q)} \\ \text{By Bellman's lemma, we get that} \\ &b_{1}(t) \leq \beta^{0} \|f_{1}(\cdot,\cdot)\|_{L^{1}(Q)} \exp(\beta^{0}T) \\ \text{We define a mapping:} \\ &F: X \to X, \\ &F(p,c_{0},c_{e}) = (F_{1}(p,c_{0},c_{e}),...,F_{5}(p,c_{0},c_{e})), \\ \text{where} \\ &F_{1}(p,c_{0},c_{e}) = p_{1}(s,t), F_{2}(p,c_{0},c_{e}) = p_{2}(s,t), \\ &F_{3}(p,c_{0},c_{e}) = c_{10}(t), F_{4}(p,c_{0},c_{e}) = c_{20}(t), \\ &F_{5}(p,c_{0},c_{e}) = c_{e}(t), \\ \text{and } F(p,c_{0},c_{e}) |(s,t) ds \\ &\leq \int_{0}^{\Gamma_{1}(t)} \Big[ b_{1}(t-\Gamma_{1}(s)) \\ &+ \int_{0}^{\Gamma_{1}(s)} f_{1}(\Gamma_{1}^{-1}(\tau),c_{10}(t+\tau-\Gamma_{1}(s))) d\tau \Big] ds \\ &\leq \int_{0}^{t} b_{1}(\sigma) d\sigma + \|f_{1}(\cdot,\cdot)\|_{L^{1}(Q)} \\ &\leq \beta^{0}T \|f_{1}(\cdot,\cdot)\|_{L^{1}(Q)} \exp(\beta^{0}T) + \|f_{1}(\cdot,\cdot)\|_{L^{1}(Q)} \coloneqq M_{1}. \\ \text{Similarly, we have} \\ &\int_{0}^{t} |F_{2}(p,c_{0},c_{e})|(s,t) ds \\ &\leq \beta^{0}T \exp\{2\lambda^{0}tM_{1} + \beta^{0}\exp(\lambda^{0}tM_{1})T\} \|f_{2}(\cdot,\cdot)\|_{L^{1}(Q)} \\ &+ \exp\{\lambda^{0}tM_{1}\} \|f_{2}(\cdot,\cdot)\|_{L^{1}(Q)} \coloneqq M_{2}. \end{split}$$

Next, we will prove that F is a contracting operator. Let  $x^1(p^1, c_0^1, c_e^1)$  and  $x^2(p^2, c_0^2, c_e^2)$  be the solutions of system (1), then we have

$$\begin{split} &\int_{0}^{l} \left| F_{1}\left(x^{1}\right) - F_{1}\left(x^{2}\right) \right| ds \\ &\leq \int_{0}^{\Gamma_{1}^{-1}(t)} \int_{0}^{l} \left| \beta_{1}\left(r, c_{10}^{1}\left(t - \Gamma_{1}\left(r\right)\right)\right) p_{1}^{1}\left(r, t - \Gamma_{1}\left(r\right)\right) \right| \\ &- \beta_{1}\left(r, c_{10}^{2}\left(t - \Gamma_{1}\left(r\right)\right)\right) p_{1}^{2}\left(r, t - \Gamma_{1}\left(r\right)\right) \right| dr ds \\ &+ \int_{0}^{\Gamma_{1}^{-1}(t)} \beta^{0} M_{1} \int_{0}^{\Gamma_{1}(s)} \left| \mu_{1}\left(\Gamma_{1}^{-1}\left(\tau\right), c_{10}^{1}\left(t + \tau - \Gamma_{1}\left(s\right)\right)\right) \right| \\ &- \mu_{1}\left(\Gamma_{1}^{-1}\left(\tau\right), c_{10}^{2}\left(t + \tau - \Gamma_{1}\left(s\right)\right)\right) \left| d\tau ds \\ &+ \int_{0}^{\Gamma_{1}^{-1}(t)} \beta^{0} M_{1} \int_{0}^{\Gamma_{1}(s)} \lambda^{0}\left(P_{2}^{1}\left(t + \tau - \Gamma_{1}\left(r\right)\right) \\ &- P_{2}^{2}\left(t + \tau - \Gamma_{1}\left(r\right)\right)\right) d\tau ds \\ &+ B \int_{0}^{\Gamma_{1}^{-1}(t)} \int_{0}^{\Gamma_{1}(s)} \int_{\tau}^{\Gamma_{1}(s)} \left(\mu_{1}\left(\Gamma_{1}^{-1}\left(\sigma\right), c_{10}^{1}\left(t + \sigma - \Gamma_{1}\left(s\right)\right)\right) \right) \\ &- \mu_{1}\left(\Gamma_{1}^{-1}\left(\sigma\right), c_{10}^{2}\left(t + \sigma - \Gamma_{1}\left(s\right)\right)\right)\right) d\sigma d\tau ds \end{split}$$

$$+B\int_{0}^{\Gamma_{1}^{-1}(t)}\int_{0}^{\Gamma_{1}(s)}\int_{\tau}^{\Gamma_{1}(s)}\lambda^{0} \left(P_{2}^{1}\left(t+\sigma-\Gamma_{1}\left(s\right)\right)\right) d\sigma d\tau ds$$

$$\leq \beta^{0}\int_{0}^{\Gamma_{1}^{-1}(t)}\int_{0}^{t}\left|p_{1}^{1}\left(r,t-\Gamma_{1}\left(r\right)\right)-p_{1}^{2}\left(r,t-\Gamma_{1}\left(r\right)\right)\right|dr ds$$

$$+\left(\beta^{0}M_{1}L_{\mu}l+Bl^{2}L_{\mu}+M_{1}L_{\beta}\right)\int_{0}^{\Gamma_{1}^{-1}(t)}\left|c_{10}^{1}\left(t+\tau-\Gamma_{1}\left(s\right)\right)\right| ds$$

$$+\left(\beta^{0}\lambda^{0}lM_{1}+B\lambda^{0}l^{2}\right)$$

$$\cdot\int_{0}^{\Gamma_{1}^{-1}(t)}\int_{0}^{\Gamma_{1}(s)}\left(p_{2}^{1}\left(\sigma,t-\Gamma_{1}\left(r\right)\right)-p_{2}^{2}\left(\sigma,t-\Gamma_{1}\left(r\right)\right)\right)d\tau ds$$

$$\leq M_{3}\left(\int_{0}^{t}\int_{0}^{l}\left|p_{1}^{1}\left(r,s\right)-p_{2}^{2}\left(r,s\right)\right|ds d\sigma$$

$$+\int_{0}^{t}\int_{0}^{l}\left|p_{2}^{1}\left(r,s\right)-p_{2}^{2}\left(r,s\right)\right|dr ds$$

$$+\int_{0}^{t}\left|c_{10}^{1}\left(s\right)-c_{10}^{2}\left(s\right)\right|ds\right), \qquad (6)$$

where  $M_3 = \max \left\{ \beta^0, \beta^0 \lambda^0 l M_1 + B \lambda^0 l^2, \\ \beta^0 M_1 L_\mu l + B l^2 L_\mu + M_1 L_\beta \right\}.$ 

Similarly, we have  $|E(x^1) - E(x^2)|$ 

$$|F_{2}(x^{1}) - F_{2}(x^{2})| \leq M_{4} \left( \int_{0}^{t} \int_{0}^{t} |p_{2}^{1}(r,s) - p_{2}^{2}(r,s)| ds d\sigma + \int_{0}^{t} |c_{20}^{1}(s) - c_{20}^{2}(s)| ds \right),$$
(7)

where 
$$M_4 = \max \exp(\lambda^0 l M_1) \{ \beta^0, \beta^0 M_1 L_\mu l + B l^2 L_\mu + M_2 L_\beta \}.$$
  
 $\left| F_3(x^1) - F_3(x^2) \right|$   
 $= \left| k \int_0^t c_e^1(s) \exp\{(s-t)(g+m)\} ds - k \int_0^t c_e^2(s) \exp\{(s-t)(g+m)\} ds \right|$   
 $\leq M_5 \int_0^t \left| c_e^1(s) - c_e^1(s) \right| ds ,$  (8)

Similarly,

$$\left| F_{4}\left(x^{1}\right) - F_{4}\left(x^{2}\right) \right|$$
  
$$\leq M_{6} \int_{0}^{t} \left| c_{e}^{1}\left(s\right) - c_{e}^{1}\left(s\right) \right| \mathrm{d}s ,$$
 (9)

where 
$$M_{5} = M_{6} = k$$
.  
 $|F_{5}(x^{1}) - F_{5}(x^{2})|$   
 $= |c_{e}(0) \exp \{-\int_{0}^{t} (k_{1}P_{1}^{1}(\tau) + k_{1}P_{2}^{1}(\tau) + h)\}$   
 $+\int_{0}^{t} (g_{1}c_{10}^{1}(s)P_{1}^{1}(s) + g_{1}c_{20}^{1}(s)P_{2}^{1}(s) + v(s))$   
 $\cdot \exp \{\int_{t}^{s} (k_{1}P_{1}^{1}(\tau) + k_{1}P_{2}^{1}(\tau) + h)d\tau\}ds$   
 $-c_{e}(0) \exp \{-\int_{0}^{t} (k_{1}P_{1}^{2}(\tau) + k_{1}P_{2}^{2}(\tau) + h)d\tau\}$   
 $-\int_{0}^{t} (g_{1}c_{10}^{2}(s)P_{1}^{2}(s) + g_{1}c_{20}^{2}(s)P_{2}^{2}(s) + v(s))$   
 $\cdot \exp \{\int_{t}^{s} (k_{1}P_{1}^{1}(\tau) + k_{1}P_{2}^{1}(\tau) + h)d\tau\}ds|$ 

$$\leq \int_{0}^{t} \left| g_{1} \left( c_{10}^{1} \left( s \right) P_{1}^{1} \left( s \right) + c_{20}^{1} \left( s \right) P_{2}^{1} \left( s \right) \right) \right. \\ \left. \left. \exp \left\{ \int_{t}^{s} k_{1} \left( P_{1}^{1} \left( \tau \right) + P_{2}^{1} \left( \tau \right) \right) d\tau \right\} \right. \\ \left. -g_{1} \left( c_{10}^{2} \left( s \right) P_{1}^{2} \left( s \right) + c_{20}^{2} \left( s \right) P_{2}^{2} \left( s \right) \right) \right. \\ \left. \left. \exp \left\{ \int_{t}^{s} k_{1} \left( P_{1}^{2} \left( \tau \right) + P_{2}^{2} \left( \tau \right) \right) d\tau \right\} \right| ds \right. \\ \left. +k_{1} \int_{0}^{t} \left( \left| P_{1}^{1} \left( \tau \right) - P_{1}^{2} \left( \tau \right) \right| + \left| P_{2}^{1} \left( \tau \right) - P_{2}^{2} \left( \tau \right) \right| \right) d\tau \right. \\ \left. + \int_{0}^{t} \left| v\left( s \right) \left( \exp \left\{ \int_{t}^{s} k_{1} \left( P_{1}^{1} \left( \tau \right) + P_{2}^{1} \left( \tau \right) \right) d\tau \right\} \right) \right| ds \right. \\ \left. - \exp \left\{ \int_{t}^{s} k_{1} \left( P_{1}^{1} \left( \tau \right) + P_{2}^{1} \left( \tau \right) \right) d\tau \right\} \right) \right| ds \right. \\ \left. \leq \left( k_{1} + g_{1} \right) \int_{0}^{t} \left( \left| P_{1}^{1} \left( s \right) - P_{1}^{2} \left( s \right) \right| + \left| P_{2}^{1} \left( s \right) - P_{2}^{2} \left( s \right) \right| \right) \right| ds \right. \\ \left. + \left( 2g_{1}k_{1}M + k_{1}v_{1} \right) \int_{0}^{t} \int_{0}^{t} \left( \left| P_{1}^{1} \left( s \right) - P_{1}^{2} \left( s \right) \right| + \left| P_{2}^{1} \left( s \right) - P_{2}^{2} \left( s \right) \right| \right) ds \right. \\ \left. + \left( 2g_{1}k_{1}M + k_{1}v_{1} \right) \int_{0}^{t} \int_{0}^{t} \left( \left| P_{1}^{1} \left( s \right) - P_{1}^{2} \left( s \right) \right| + \left| P_{2}^{1} \left( s \right) - P_{2}^{2} \left( s \right) \right| \right) ds \right. \\ \left. \left. \left. \left( 2g_{1}k_{1}M + k_{1}v_{1} \right) \int_{0}^{t} \int_{0}^{t} \left| P_{1}^{1} \left( r, s \right) - p_{1}^{2} \left( r, s \right) \right| dr ds \right. \\ \left. \left. \left. \left( 2g_{1}^{2} \left[ \int_{0}^{t} \int_{0}^{t} \left| P_{1}^{1} \left( r, s \right) - p_{1}^{2} \left( r, s \right) \right| dr ds \right. \right. \\ \left. \left. \left. \left( 2g_{1}^{2} \left[ \int_{0}^{t} \int_{0}^{t} \left| P_{1}^{1} \left( r, s \right) - p_{1}^{2} \left( r, s \right) \right| dr ds \right. \right. \right. \right.$$

$$\left. \left. \left. \left. \left( 2g_{1}^{2} \left[ \int_{0}^{t} \left| P_{1}^{1} \left( r, s \right) - p_{1}^{2} \left( r, s \right) \right| dr ds \right. \right. \right. \right. \right. \right. \right.$$

$$\left. \left. \left. \left( 2g_{1}^{2} \left[ \int_{0}^{t} \left| P_{1}^{1} \left( r, s \right) - c_{10}^{2} \left| r, s \right) \right| dr ds \right. \right. \right. \right. \right.$$

$$\left. \left. \left( 2g_{1}^{2} \left[ \int_{0}^{t} \left| P_{1}^{1} \left( r, s \right) - c_{10}^{2} \left| r, s \right) \right] \right] \right. \right. \right. \right. \right. \right.$$

where  $M_7 = \max \{k_1 + g_1 + k_1 v_1 T + 2k_1 g_1 M T, g_1 M\}$ ,  $M = \max \{M_1, M_2\}$ . We define an equivalent norm in space X by

We define an equivalent norm in space X by  

$$\|(p,c_{0},c_{e})\|_{*} = \operatorname{Ess\,sup}_{t\in[0,T]} e^{-\lambda t} \left\{ \sum_{i=1}^{2} \int_{0}^{t} p_{i}(s,t) dt + \sum_{i=1}^{2} c_{i0}(t) |+|c_{e}(t)| \right\},$$
where  $\lambda > 0$  large enough. By (6)-(10), we get  

$$\|F(x^{1}) - F(x^{2})\|_{*}$$

$$= \|(F_{1}(x^{1}) - F_{1}(x^{2}), F_{2}(x^{1}) - F_{2}(x^{2}), \dots, F_{5}(x^{1}) - F_{5}(x^{2}))\|_{*}$$

$$\leq M_{8} \operatorname{Ess\,sup}_{t\in[0,T]} e^{-\lambda t} \int_{0}^{t} \left\{ \sum_{i=1}^{2} \int_{0}^{t} |p_{i}^{1}(r,s) - p_{i}^{2}(r,s)| dr$$

$$+ \sum_{i=1}^{2} |c_{i0}^{1}(s) - c_{i0}^{2}(s)| + |c_{e}^{1}(s) - c_{e}^{2}(s)| \right\} ds$$

$$= M_{8} \operatorname{Ess\,sup}_{t\in[0,T]} e^{-\lambda t} \int_{0}^{t} e^{\lambda s} \left\{ e^{-\lambda s} \left[ \sum_{i=1}^{2} \int_{0}^{t} |p_{i}^{1}(r,s) - p_{i}^{2}(r,s)| dr$$

$$+ \sum_{i=1}^{2} |c_{i0}^{1}(s) - c_{i0}^{2}(s)| + |c_{e}^{1}(s) - c_{e}^{2}(s)| \right\} ds$$

$$\leq M_{8} \|x^{1} - x^{2}\|_{*} \operatorname{Ess\,sup}_{t\in[0,T]} \left\{ e^{-\lambda t} \int_{0}^{t} e^{\lambda s} ds \right\}$$

$$\leq \frac{M_{8}}{\lambda} \|x^{1} - x^{2}\|_{*},$$

where  $M_8 = \max \{M_3, M_4, M_5, M_6, M_7\}$ . It obvious that for any  $\lambda > M_8$ , then F is a contraction on the space of  $(X, \|\cdot\|_*)$  and owns a unique fixed point. Namely,  $(p, c_0, c_e)$  is the solution of the state system (1).

**Lemma 3.1.3** For any  $(u^i, v^i) \in \Omega$ , the state system (1)

exists a unique non-negative and bounded solution, where  $|p_i(s,t)| \le C_1$ . Moreover, there exists  $K_i > 0$ , such that

$$\begin{split} &\sum_{i=1}^{2} \left\| p_{i}^{1} - p_{i}^{2} \right\|_{L^{\infty}\left(0,T;L^{1}\left(0,I\right)\right)} + \sum_{i=1}^{2} \left\| c_{i0}^{1} - c_{i0}^{2} \right\|_{L^{\infty}\left(0,T\right)} + \left\| c_{e}^{1} - c_{e}^{2} \right\|_{L^{\infty}\left(0,T\right)} \\ &\leq K_{1}T \left( \sum_{i=1}^{2} \left\| u_{i}^{1} - u_{i}^{2} \right\|_{L^{\infty}\left(0,T;L^{1}\left(0,I\right)\right)} + \left\| v^{1} - v^{2} \right\|_{L^{\infty}\left(0,T\right)} \right), \\ &\sum_{i=1}^{2} \left\| p_{i}^{1} - p_{i}^{2} \right\|_{L^{1}\left(Q\right)} + \sum_{i=1}^{2} \left\| c_{i0}^{1} - c_{i0}^{2} \right\|_{L^{1}\left(0,T\right)} + \left\| c_{e}^{1} - c_{e}^{2} \right\|_{L^{1}\left(0,T\right)} \\ &\leq K_{2}T \left( \sum_{i=1}^{2} \left\| u_{i}^{1} - u_{i}^{2} \right\|_{L^{1}\left(Q\right)} + \left\| v^{1} - v^{2} \right\|_{L^{1}\left(0,T\right)} \right). \end{split}$$

The proof process of lemma 3.1.3 is similar to that of Theorem 3.1.2, which is omitted here.

#### 3.2 Optimality conditions

In this section, we consider the adjoint system of (1) and establish first-order necessary conditions for optimal harvesting of (2).

**Lemma 3.2.1** Let  $(p^*, c_0^*, c_e^*)$  be the solution of system (1) corresponding to  $(u^*, v^*) \in \Omega$ . For any  $(v_1, v_2) \in T_{\Omega}(u^*, v^*)$ ,  $u^* = (u_1^*, u_2^*)$ ,  $v_1 = (v_{11}, v_{21})$ , such that  $(u^* + \varepsilon v_1, +v^* + \varepsilon v_2) \in \Omega$  for sufficiently small  $\varepsilon > 0$ , we obtain

$$\frac{1}{\varepsilon} \left( p_i^{\varepsilon}(s,t) - p_i^{*}(s,t) \right) \to z_i(s,t),$$
  
$$\frac{1}{\varepsilon} \left( c_{i0}^{\varepsilon}(t) - c_{i0}^{*}(t) \right) \to z_{i+2}(t),$$
  
$$\frac{1}{\varepsilon} \left( c_e^{\varepsilon}(t) - c_e^{*}(t) \right) \to z_5(t),$$

as  $\varepsilon \to 0$ , where  $(p^{\varepsilon}, c_0^{\varepsilon}, c_e^{\varepsilon})$  is the solution of (1) corresponding to  $(u^* + \varepsilon v_1, v^* + \varepsilon v_2) \in \Omega$  and  $(z_1, z_2, z_3, z_4, z_5)$  is the solution of the following system

$$\begin{cases} \frac{\partial z_{1}}{\partial t} + g_{1} \frac{\partial z_{1}}{\partial s} = -\left[\mu_{1}\left(s, c_{10}^{*}(t)\right) + \lambda_{1}P_{2}^{*}(t) + g_{1s}(s) + u_{1}^{*}(s,t)\right] z_{1}(s,t) - \int_{0}^{t} \lambda_{1}p_{1}^{*}(s,t)z_{2}(s,t) ds \\ - \frac{\partial \mu_{1}\left(s, c_{10}^{*}(t)\right)}{\partial c_{10}} p_{1}^{*}(s,t)z_{3}(t) - v_{11}p_{1}^{*}(s,t), \\ \frac{\partial z_{2}}{\partial t} + g_{2} \frac{\partial z_{2}}{\partial s} = -\left[\mu_{2}\left(s, c_{20}^{*}(t)\right) - \lambda_{2}P_{1}^{*}(t) + g_{2s}(s) + u_{2}^{*}(s,t)\right] z_{2}(s,t) + \int_{0}^{t} \lambda_{2}p_{2}^{*}(s,t)z_{1}(s,t) ds \\ - \frac{\partial \mu_{2}\left(s, c_{20}^{*}(t)\right)}{\partial c_{20}} p_{1}^{*}(s,t)z_{4}(t) - v_{21}p_{2}^{*}(s,t), \\ \frac{dz_{3}}{dt} = kz_{5}(t) - gz_{3}(t) - mz_{3}(t), \\ \frac{dz_{4}}{dt} = kz_{5}(t) - gz_{4}(t) - mz_{4}(t), \\ \frac{dz_{5}}{dt} = -k_{1}c_{e}^{*}(t)(Z_{1}(t) + Z_{2}(t)) + g_{1}c_{10}^{*}(t)Z_{1}(t) \\ + g_{2}c_{20}^{*}(t)Z_{2}(t) + g_{1}\left(P_{1}^{*}(t)z_{3}(t) + P_{2}^{*}(t)z_{4}(t)\right) \\ - \left[k_{1}\left(P_{1}^{*}(t) + P_{2}^{*}(t)\right) + h\right]z_{5}(t) + v_{2}, \\ g_{i}(0)z_{i}(0,t) = \int_{0}^{t} \beta_{i}\left(s, c_{i0}^{*}(t)\right)z_{i}(t) ds \\ + \int_{0}^{t} \frac{\partial \beta\left(s, c_{i0}^{*}(t)\right)}{\partial c_{i0}} p_{i}^{*}(s,t)z_{i+2}(t) ds, \\ z_{i}(s,t) = z_{i}(s,t+T), \\ z_{i+2}(0) = z_{5}(0) = 0, \\ Z_{i}(t) = \int_{0}^{t} z_{i}(s,t) ds, P_{i}^{*}(t) = \int_{0}^{t} p_{i}^{*}(s,t) ds. \end{cases}$$

**Theorem 3.2.2** Let  $(u^*, v^*)$  be an optimal policy for the optimal harvesting problem of (1)-(2). Then

$$u_{i}^{*}(s,t) = L_{i}\left(\frac{\left[w_{i}(s,t) - q_{i}(s,t)\right]p_{i}^{*}}{c_{i}}\right), i = 1, 2,$$
$$v^{*}(t) = L_{3}\left(\frac{q_{5}(t)}{c_{3}}\right),$$

in which  $L_i$  are given by

$$L_{j}(\eta)(s,t) = \begin{cases} 0, & \eta(s,t) < 0, \\ \eta(s,t), & 0 \le \eta(s,t) \le H_{j}, j = 1, 2, 3 \\ H_{j}, & \eta(s,t) > H_{j}, \end{cases}$$

where  $(q_1, q_2, q_3, q_4, q_5)$  is the solution of the following adjoint system (12):

$$\begin{split} \frac{\partial q_1}{\partial t} + g_1 \frac{\partial q_1}{\partial s} &= \left[ \mu_1 \left( s, c_{10}^* \left( t \right) \right) + \lambda_1 \left( s, t \right) P_2^* \left( t \right) \right. \\ &+ u_1^* \left( s, t \right) \right] q_1 \left( s, t \right) + \left( k_1 c_e^* \left( t \right) - g_1 c_0^* \left( t \right) \right) q_5 \left( t \right) \right. \\ &- q_1 \left( 0, t \right) \beta_1 \left( s, c_{10}^* \left( t \right) \right) + w_1 \left( s, t \right) u_1^* \left( s, t \right), \\ \frac{\partial q_2}{\partial t} + g_2 \frac{\partial q_2}{\partial s} &= \left[ \mu_2 \left( s, c_{20}^* \left( t \right) \right) - \lambda_2 \left( s, t \right) P_1^* \left( t \right) \right. \\ &+ u_2^* \left( s, t \right) \right] q_2 \left( s, t \right) + \left( k_1 c_e^* \left( t \right) - g_1 c_{20}^* \left( t \right) \right) q_5 \left( t \right) \right. \\ &- q_2 \left( 0, t \right) \beta_2 \left( s, c_{20}^* \left( t \right) \right) + w_2 \left( s, t \right) u_2^* \left( s, t \right), \\ \frac{d q_3 \left( t \right)}{d t} &= \int_0^t \frac{\partial \mu_1 \left( s, c_{10}^* \left( t \right) \right)}{\partial c_{10}} p_1^* \left( s, t \right) q_5 \left( t \right) \\ &- q_1 \left( 0, t \right) \int_0^t \frac{\partial \beta_1 \left( s, c_{10}^* \left( t \right) \right)}{\partial c_{10}} p_1^* \left( s, t \right) ds, \\ &+ \left( g + m \right) q_3 \left( t \right) - g_1 P_1^* \left( t \right) q_5 \left( t \right) \\ &- q_1 \left( 0, t \right) \int_0^t \frac{\partial \beta_1 \left( s, c_{10}^* \left( t \right) \right)}{\partial c_{10}} p_1^* \left( s, t \right) ds, \\ &+ \left( g + m \right) q_4 \left( t \right) - g_1 P_2^* \left( t \right) q_5 \left( t \right) \\ &- q_2 \left( 0, t \right) \int_0^t \frac{\partial \beta_2 \left( s, c_{20}^* \left( t \right) \right)}{\partial c_{20}} p_2^* \left( s, t \right) ds, \\ &+ \left( k_1 P_1^* \left( t \right) + k_1 P_2^* \left( t \right) + h \right) q_5 \left( t \right), \\ &q_i \left( l, t \right) = 0, \\ &q_i \left( s, t \right) = q_i \left( s, t + T \right), \\ &q_{i+1} \left( T \right) = q_{i+2} \left( T \right) = 0, i = 1, 2. \end{split}$$

**Proof** For any given element of the tangent cone  $(v_1, v_2) \in T_{\Omega}(u^*, v^*)$ , we have  $(u^* + \varepsilon v_1, v^* + \varepsilon v_2) \in \Omega$  for any sufficiently small  $\varepsilon > 0$ . Then, due to the optimality of  $(u^*, v^*)$ , it is derived that

$$\begin{split} &\sum_{i=1}^{2} \int_{0}^{T} \int_{0}^{l} w_{i}(s,t) (u_{i}^{*} + \varepsilon v_{i1}) p_{i}^{\varepsilon}(s,t) ds dt \\ &- \frac{1}{2} \sum_{i=1}^{2} \int_{0}^{T} \int_{0}^{l} c_{i} (u_{i}^{*} + \varepsilon v_{i1}) ds dt - \frac{1}{2} \int_{0}^{T} c_{3} (v^{*} + \varepsilon v_{2}) dt \\ &\leq \sum_{i=1}^{2} \int_{0}^{T} \int_{0}^{l} w_{i}(s,t) u_{i}^{*}(s,t) p_{i}^{*}(s,t) ds dt \\ &- \frac{1}{2} \sum_{i=1}^{2} \int_{0}^{T} \int_{0}^{l} c_{i} [u_{i}^{*}(s,t)]^{2} ds dt - \frac{1}{2} \int_{0}^{T} c_{3} [v^{*}(t)]^{2} dt \, . \end{split}$$

By Lemma 3.2.1, it follows that

$$\sum_{i=1}^{2} \int_{0}^{T} \int_{0}^{I} w_{i}(s,t) u_{i}^{*}(s,t) z_{i}(s,t) ds dt$$
  
+ 
$$\sum_{i=1}^{2} \int_{0}^{T} \int_{0}^{I} \left[ w_{i}(s,t) p_{i}^{*}(s,t) - c_{i} u_{i}^{*}(s,t) \right] v_{i1}(s,t) ds dt$$

$$-\int_{0}^{T} c_{3} v^{*}(t) v_{2}(t) \mathrm{d}t \leq 0 , \qquad (13)$$

Multiplying the first five equations in (11) by  $q_1(s,t), q_2(t), ..., q_5(t)$  respectively, then integrating on Q and (0,T), combining (13), we get

$$\sum_{i=1}^{2} \int_{0}^{T} \int_{0}^{l} w_{i}(s,t) u_{i}^{*}(s,t) z_{i}(s,t) ds dt$$
  
=  $-\sum_{i=1}^{2} \int_{0}^{T} \int_{0}^{l} v_{i1}(s,t) p_{i}^{*}(s,t) q_{i}(s,t) ds dt$   
+  $\int_{0}^{T} v_{2}(t) q_{5}(t) dt$ . (14)

Substituting (14) into (13) gives that

$$\begin{split} \sum_{i=1}^{2} \int_{0}^{T} \int_{0}^{t} \left( \left( w_{i} - q_{i} \right) p_{i}^{*} - c_{i} u_{i}^{*} \right) v_{i1} \mathrm{d}s \, \mathrm{d}t \\ + \int_{0}^{T} \left( -c_{3} v^{*} + q_{5} \right) v_{2} \mathrm{d}t &\leq 0 , \\ \text{for every } (v_{1}, v_{2}) \in T_{\Omega} \left( u^{*}, v^{*} \right) , \quad \text{thus} \\ \left( \left( w_{i} - q_{i} \right) p_{i}^{*} - c_{i} u^{*}, -c_{3} v^{*} + q_{5} \right) \in N_{\Omega} \left( u^{*}, v^{*} \right) , \quad \text{where} \\ N_{\Omega} \left( u^{*}, v^{*} \right) \text{ is normal cone of } \Omega \text{ at } \left( u^{*}, v^{*} \right). \end{split}$$

**Lemma 3.2.3** For any  $(u^i, v^i) \in \Omega$ , i = 1, 2, The adjoint system (12) exists a unique bounded solution, where  $|q_i| \le C_2$ . Moreover, there exists  $K_3 > 0$ , such that

$$\begin{split} &\sum_{i=1}^{2} \|q_{i}^{1}-q_{i}^{2}\|_{L^{\infty}(\mathcal{Q})} + \sum_{\xi=3}^{4} \|q_{\xi}^{1}-c_{\xi}^{2}\|_{L^{\infty}(0,T)} + \|q_{5}^{1}-q_{5}^{2}\|_{L^{\infty}(0,T)} \\ &\leq K_{3}T \left(\sum_{i=1}^{2} \|u_{i}^{1}-u_{i}^{2}\|_{L^{\infty}(\mathcal{Q})} + \|v^{1}-v^{2}\|_{L^{\infty}(0,T)}\right). \end{split}$$

#### 3.3 Existence of optimal policy

In this section, the existence of optimal policy will be established. First, define the embedding mapping  $\tilde{J}(u,v)$ 

in 
$$[L^{1}(Q)]^{2} \times L^{1}(0,T)$$
 as follows  
 $\tilde{J}(u,v) = \begin{cases} J(u,v), & (u,v) \in \Omega \\ -\infty, & (u,v) \notin \Omega \end{cases}$ 

**Lemma 3.3.1** The functional  $\widetilde{J}(u,v)$  is upper semicontinuous in  $\left[L^{1}(Q)\right]^{2} \times L^{1}(0,T)$ .

**Proof** Suppose  $(u^n, v^n) \rightarrow (u, v)$  as  $n \rightarrow \infty$ ,  $(p^n, c_0^n, c_0^n)$ and  $(p, c_0, c_e)$  be solutions of (1) corresponding to  $(u^n, v^n)$  and (u, v), respectively. According to the Riesz theorem, there exists a subsequence (still denoted by  $(u^n, v^n)$ ), it follows that

$$\begin{bmatrix} u^{n}(s,t)^{2} \to u^{2}(s,t), \text{ a.e. } (s,t) \in Q, \text{ as } n \to \infty. \quad (15) \\ \begin{bmatrix} u^{n}(s,t)^{2} \to u^{2}(s,t), \text{ a.e. } (s,t) \in Q, \text{ as } n \to \infty. \end{bmatrix}$$

 $[v^n(t)] \rightarrow v^2(t)$ , a.e.  $t \in (0,T)$ , as  $n \rightarrow \infty$ . (16) From (15) and (16), using Lebesgue dominated convergence theorem implies that

$$\begin{split} \lim_{n \to \infty} \int_{0}^{T} \int_{0}^{l} \left[ u_{i}^{n}\left(s,t\right) \right]^{2} \mathrm{d}s \mathrm{d}t &= \int_{0}^{T} \int_{0}^{l} \left[ u_{i}\left(s,t\right) \right]^{2} \mathrm{d}s \mathrm{d}t \;, \\ \lim_{n \to \infty} \int_{0}^{T} \left[ v^{n}(t) \right]^{2} \mathrm{d}t &= \int_{0}^{T} v^{2}(t) \mathrm{d}t \;. \end{split}$$
By Lemma 3.1.3, we have
$$\left| \int_{0}^{T} \int_{0}^{l} w_{i}\left(s,t\right) u_{i}^{n}\left(s,t\right) p_{i}^{n}\left(s,t\right) \mathrm{d}s \mathrm{d}t - \int_{0}^{T} \int_{0}^{l} w_{i}\left(s,t\right) u_{i}\left(s,t\right) p_{i}\left(s,t\right) \mathrm{d}s \mathrm{d}t \right| \\ &\leq \int_{0}^{T} \int_{0}^{l} w_{i}\left(s,t\right) p_{i}^{n}\left(s,t\right) \left| u_{i}^{n}\left(s,t\right) - u\left(s,t\right)_{i} \right| \mathrm{d}s \mathrm{d}t \\ &+ \int_{0}^{T} \int_{0}^{l} w_{i}\left(s,t\right) u_{i}\left(s,t\right) \left| p_{i}^{n}\left(s,t\right) - p_{i}\left(s,t\right) \right| \mathrm{d}s \mathrm{d}t \\ &\leq M \overline{w} \left\| u_{i}^{n} - u_{i} \right\|_{L^{1}(Q)} + \overline{w} H \left\| p^{n} - p \right\|_{L^{1}(Q)} \\ &\leq M \overline{w} \left\| u_{i}^{n} - u_{i} \right\|_{L^{1}(Q)} + \overline{w} H K_{2} T \left( \left\| u_{i}^{n} - u_{i} \right\|_{L^{1}(Q)} + \left\| v^{n} - v \right\|_{L^{1}(0,T)} \right), \end{aligned}$$
we get that
$$\lim_{n \to \infty} \int_{0}^{T} \int_{0}^{l} w_{i}\left(s,t\right) u_{i}^{n}\left(s,t\right) u_{i}^{n}\left(s,t\right) p_{i}^{n}\left(s,t\right) \mathrm{d}s \mathrm{d}t \end{aligned}$$

$$\lim_{n \to \infty} \int_0^T \int_0^t w_i(s,t) u_i^n(s,t) p_i^n(s,t) \mathrm{d}s \mathrm{d}t$$
$$= \int_0^T \int_0^t w_i(s,t) u_i(s,t) p_i(s,t) \mathrm{d}s \mathrm{d}t .$$

Therefore, we have proved that

$$\lim_{n\to\infty}\sup_{\left(u^n,v^n\right)\in\Omega}\widetilde{J}\left(u^n,v^n\right)\leq\widetilde{J}\left(u,v\right)$$

**Theorem 3.3.2** If  $\left[c_1^{-1}T(\bar{w}K_1+C_2K_1+C_1K_3)+c_2^{-1}K_3T\right] \le 1$ , the optimal harvesting problem (1) and (2) exist a unique optimal policy.

Proof According to Lemma 3.3.1 and Ekeland's variational principle, for each  $\varepsilon > 0$  , there is a  $(u^{\varepsilon}, v^{\varepsilon}) \in \Omega$  such that

$$\widetilde{J}(u^{\varepsilon}, v^{\varepsilon}) \geq \sup_{(u,v)\in\Omega} \widetilde{J}(u,v) - \varepsilon , \qquad (17)$$
$$\widetilde{J}(u^{\varepsilon}, v^{\varepsilon}) \geq \sup_{(u,v)\in\Omega} \left\{ \widetilde{J}(u,v) - \sqrt{\varepsilon} \left( \sum_{i=1}^{2} \|u_{i}^{\varepsilon} - u_{i}\|_{L^{1}(Q)} + \|v^{\varepsilon} - v\|_{L^{1}(0,T)} \right) \right\} . (18)$$

By Theorem 3.2.2, we have

$$\begin{split} \left( w_i - q_i^{\varepsilon} \right) p_i^{\varepsilon} - c_i u_i^{\varepsilon} + \sqrt{\varepsilon} \theta_i^{\varepsilon} \in N_{\Omega} \left( u_i^{\varepsilon} \right), & i = 1, 2, \\ q_5^{\varepsilon} - c_3 v^{\varepsilon} + \sqrt{\varepsilon} \theta_3^{\varepsilon} \in N_{\Omega} \left( v^{\varepsilon} \right). \end{split}$$

Consequently

$$\begin{pmatrix} u^{\varepsilon}, v^{\varepsilon} \end{pmatrix} = \left( L_1 \left( \frac{\left( w_1 - q_1^{\varepsilon} \right) p_1^{\varepsilon}}{c_1} + \frac{\sqrt{\varepsilon} \theta_1^{\varepsilon}}{c_1} \right), \\ L_2 \left( \frac{\left( w_2 - q_2^{\varepsilon} \right) p_2^{\varepsilon}}{c_2} + \frac{\sqrt{\varepsilon} \theta_2^{\varepsilon}}{c_2} \right), L_3 \left( \frac{q_5^{\varepsilon}}{c_3} + \frac{\sqrt{\varepsilon} \theta_3^{\varepsilon}}{c_3} \right) \right),$$

where  $\theta_1^{\varepsilon}, \theta_2^{\varepsilon} \in L^{\infty}(Q)$ ,  $\theta_3^{\varepsilon} \in L^{\infty}(0,T)$  with  $|\theta_1^{\varepsilon}| \le 1$ ,  $\left| \theta_2^{\varepsilon} \right| \leq 1, \left| \theta_3^{\varepsilon} \right| \leq 1,$ 

First, we prove the uniqueness of optimal harvesting by the contraction mapping principle.

$$F(u,v) = \left(L_{1}\left(\frac{(w_{1}-q_{1})p_{1}}{c_{1}}\right), L_{2}\left(\frac{(w_{2}-q_{2})p_{2}}{c_{2}}\right), L_{3}\left(\frac{q_{5}}{c_{3}}\right)\right).$$

Therefore, we have  

$$\begin{aligned} & \left|F(u^{1},v^{1})-F(u^{2},v^{2})\right| \\ \leq c_{1}^{-1}\left|\left(w_{1}-q_{1}^{1}\right)p_{1}^{1}-\left(w_{2}-q_{1}^{2}\right)p_{1}^{2}\right| \\ & +c_{2}^{-1}\left|\left(w_{2}-q_{2}^{1}\right)p_{2}^{1}-\left(w_{2}-q_{2}^{2}\right)p_{2}^{2}\right|+c_{3}^{-1}\left|q_{5}^{1}-q_{5}^{2}\right|. \end{aligned}$$
Combining Lemma 3.1.3 with Lemma 3.2.3, we get that
$$\begin{aligned} & \left\|F(u^{1},v^{1})-F(u^{2},v^{2})\right\| \\ \leq & \left[T\left(c_{1}^{-1}+c_{2}^{-1}\right)\left(\overline{w}K_{1}+C_{2}K_{1}+C_{1}K_{3}\right)+c_{3}^{-1}K_{3}T\right] \\ & \cdot \left(\sum_{i=1}^{2}\left\|u_{i}^{1}-u_{i}^{2}\right\|_{L^{\infty}(Q)}+\left\|v^{1}-v^{2}\right\|_{L^{\infty}(0,T)}\right). \end{aligned}$$
If  $\left[T\left(c_{1}^{-1}+c_{2}^{-1}\right)\left(\overline{w}K_{1}+C_{2}K_{1}+C_{1}K_{3}\right)+c_{3}^{-1}K_{3}T\right] < 1$ , the mapping  $F$  is a contraction and has a unique fixed point  $(\overline{u},\overline{v}) \in \Omega$ . Then the Theorem 3.2.2 implies that any optimal control  $(u^{*},v^{*})$ , if exists must be a fixed point of the mapping  $F$ . Next we will show that  $(\overline{u},\overline{v}) \in \Omega$  is the optimal

policy. Since  

$$\begin{aligned} & \|F(u^{\varepsilon}, v^{\varepsilon}) - (u^{\varepsilon}, v^{\varepsilon})\| \\ &= \left\| \left( L_{1} \left( \frac{(w_{1} - q_{1}^{\varepsilon}) p_{1}^{\varepsilon}}{c_{1}} \right), L_{2} \left( \frac{(w_{2} - q_{2}^{\varepsilon}) p_{2}^{\varepsilon}}{c_{2}} \right), L_{3} \left( \frac{q_{5}^{\varepsilon}}{c_{3}} \right) \right) \\ &- \left( L_{1} \left( \frac{(w_{1} - q_{1}^{\varepsilon}) p_{1}^{\varepsilon}}{c_{1}} + \frac{\sqrt{\varepsilon} \theta_{1}}{c_{1}} \right), L_{2} \left( \frac{(w_{2} - q_{2}^{\varepsilon}) p_{2}^{\varepsilon}}{c_{2}} + \frac{\sqrt{\varepsilon} \theta_{2}}{c_{2}} \right), \\ &L_{3} \left( \frac{q_{5}^{\varepsilon}}{c_{3}} + \frac{\sqrt{\varepsilon} \theta_{3}}{c_{3}} \right) \right) \right\| \\ &\leq c_{1}^{-1} \sqrt{\varepsilon} \left\| \theta_{1}^{\varepsilon} \right\| + c_{2}^{-1} \sqrt{\varepsilon} \left\| \theta_{2}^{\varepsilon} \right\| + c_{3}^{-1} \sqrt{\varepsilon} \left\| \theta_{3}^{\varepsilon} \right\| \\ &\leq \sqrt{\varepsilon} \left( c_{1}^{-1} + c_{2}^{-1} + c_{3}^{-1} \right). \\ &\text{and } F(\overline{u}, \overline{v}) = (\overline{u}, \overline{v}), \text{ it follows that} \\ & \left\| (\overline{u}, \overline{v}) - (u^{\varepsilon}, v^{\varepsilon}) \right\| \\ &\leq \left[ F(\overline{u}, \overline{v}) - F(u^{\varepsilon}, v^{\varepsilon}) \right] + \left\| F(u^{\varepsilon}, v^{\varepsilon}) - (u^{\varepsilon}, v^{\varepsilon}) \right\| \\ &\leq \left[ T\left( c_{1}^{-1} + c_{2}^{-1} \right) (\overline{w}K_{1} + C_{2}K_{1} + C_{1}K_{3}) + c_{3}^{-1}K_{3}T \right] \\ &\cdot \left( \sum_{i=1}^{2} \left\| \overline{u}_{i} - u_{i}^{\varepsilon} \right\|_{L^{\alpha}(Q)} + \left\| \overline{v} - v^{\varepsilon} \right\|_{L^{\alpha}(0T)} \right) + \sqrt{\varepsilon} \left( c_{1}^{-1} + c_{2}^{-1} + c_{3}^{-1} \right) \\ &\leq \left( \max \left\{ K_{3}, K_{4} \right\} \right) T\left( c_{1}^{-1} + c_{2}^{-1} + c_{3}^{-1} \right) \\ &\cdot \left( \sum_{i=1}^{2} \left\| \overline{u}_{i} - u_{i}^{\varepsilon} \right\|_{L^{\alpha}(Q)} + \left\| \overline{v} - v^{\varepsilon} \right\|_{L^{\alpha}(0T)} \right) + \sqrt{\varepsilon} \left( c_{1}^{-1} + c_{2}^{-1} + c_{3}^{-1} \right), \\ \text{ that is} \end{aligned}$$

$$\sum_{i=1}^{2} \left\| \overline{u_{i}} - u_{i}^{\varepsilon} \right\|_{L^{\infty}(\mathcal{Q})} + \left\| \overline{v} - v^{\varepsilon} \right\|_{L^{\infty}(0,T)}$$

$$\leq \frac{\sqrt{\varepsilon} \left( c_{1}^{-1} + c_{2}^{-1} + c_{3}^{-1} \right)}{\left( 1 - \max\left\{ K_{3}, K_{4} \right\} \right) T \left( c_{1}^{-1} + c_{2}^{-1} + c_{3}^{-1} \right)} .$$

where  $\overline{w}K_1 + C_2K_1 + C_1K_3 = K_4$ . Therefore  $(u^{\varepsilon}, v^{\varepsilon}) \rightarrow (\overline{u}, \overline{v})$  as  $\varepsilon \rightarrow 0$ . From Lemma

1.

3.3.1 and inequality (17) it follows that  $\widetilde{J}(\overline{u},\overline{v}) = \sup_{(u,v)\in\Omega} \widetilde{J}(u,v),$ 

which implies that  $(\overline{u}, \overline{v}) \in \Omega$  is the optimal policy.

## 4 Conclusion

In this paper, an optimal development model of a periodic population system dependent on sized-structure in a polluted environment is established and analyzed. If all the parameters in the optimal control problem (1) and (2) are periodic functions related to time, then with the increase of time, the total population resources will show periodic changes. From the Theorem 3.2.2 and 3.3.2, the optimal harvesting problem (1) and (2) have at least one optimal harvesting policy, and the Theorem 3.2.2 shows that the optimal harvesting policy has a Bang-Bang structure. The optimal harvesting policy in Theorem 3.2.2 provides a threshold condition for optimal harvesting problems (1) and (2):

$$(w_i - q_i) p_i^* - c_i u^*, -c_3 v^* + q_5, i = 1, 2$$
.

If  $(w_i - q_i)p_i^* - c_iu^* < 0, q_5 - c_3v^* < 0$ , that is, when the economic value of the population is relatively low, then the optimal harvesting policy is equal to its minimum value 0. If  $(w_i - q_i)p_i^* - c_iu^* > 0, q_5 - c_3v^* > 0$ , that is, when the economic value of the population is relatively high, then the optimal harvesting policy is equal to its maximum value  $H_j, j = 1, 2, 3$ . This result has obvious practical significance.

In practical applications, the parameter values in models (1) and (2) are fitted by observation data. Then combined with (2), (3) and (12) to calculate the optimal harvest policy and the optimal population density. Furthermore, the optimal index s can be obtained through numerical calculation, which provides a theoretical basis for the development and utilization of population resources.

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