# $I_{1/2}$ regularization for wavelet frames based few-view CT reconstruction

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**Abstract.** Reducing the radiation exposure in computed tomography (CT) is always a significant research topic in radiology. Image reconstruction from few-view projection is a reasonable and effective way to decrease the number of rays to lower the radiation exposure. But how to maintain high image reconstruction quality while reducing radiation exposure is a major challenge. To solve this problem, several researchers are absorbed in  $l_0$  or  $l_1$  regularization based optimization models to deal with it. However, the solution of  $l_1$  regularization based optimization model is not sparser than that of  $l_{1/2}$  or  $l_0$  regularization, and solving the  $l_0$  regularization is more difficult than solving the  $l_{1/2}$  regularization. In this paper, we develop  $l_{1/2}$  regularization for wavelet frames based image reconstruction model to research the few-view problem. First, the existence of the solution of the corresponding model is demonstrated. Second, an alternate direction method (ADM) is utilized to separate the original problem into two subproblems, where the former subproblem about the image is solved using the idea of the proximal mapping, the simultaneous algebraic reconstruction technique (SART) and the projection and contraction (PC) algorithm, and the later subproblem about the wavelet coefficients is solved using the half thresholding (HT) algorithm. Furthermore, the convergence analysis of our method is given by the simulated implementions. Simulated and real experiments confirm the effectiveness of our method.

# **1** Introduction

Nowadays, X-ray computed tomography (CT) has been extensively applied in medical diagnosis, biomedical research, non-destructive testing and so forth [1]. However, the inherent radiation dose of CT may induce cancer and other diseases in medical diagnosis [2,3]. The reduction of X-ray radiation is a more and more urgent issue. Generally, there are two strategies to reduce radiation dose: one is to lower the tube current or voltage values in data acquisition protocols, and the other is to decrease the number of the X-ray attenuation measurements through an object to be reconstructed. The former situation often introduces excessive noise into the projection data [4,5]. The latter situation necessarily results in insufficient projection data, which leads to few-view reconstruction [8-11], limited-angle reconstruction [8,13,14], etc.. In this paper, we focus on the few-view reconstruction within short-scan. Iconically, The sketch map of the scanning geometry for the few-view reconstruction problem within short-scan can be presented as Figure 1, where the red point denotes the sampled X-ray source.



**Figure 1.** The sketch map of the scanning geometry for the few-view reconstruction within short-scan.

Restricted by the noise and the number of the

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projection data, the image reconstruction problem is usually an ill-posed inverse problem from a mathematical point of view [1] formulated as:

$$Ax + e = b. \tag{1}$$

where x denotes the discrete attenuation coefficients of the object to be reconstructed (that is the image); A is the system matrix; e represents the noise with level (that is  $||e||_2 \le t$ ) and b is the measurement of the calibrated and log-transformed projection data. Generally, the solution of (1) can be found by minimizing the following optimization problem:

$$\min_{x} \left\| Ax - b \right\|_{D}^{2}.$$
 (2)

where  $||x||_D^2 = x^T Dx$  and D is a positive definite diagonal matrix. <sup>*T*</sup> denotes the transposition.

When the projection data is complete, the commercial Filtered Back-Projection (FBP) [6] method is а common choice for two-dimensional reconstruction. However, if the projection data is incomplete or contains high noise, the reconstructed images using commercial FBP method will suffer from artifacts and noise. Fortunately, the iterative algorithms (such as the simultaneous algebraic reconstruction technique (SART) [6,7]) may perform better than FBP method. SART can suppress the noise well when the projection data with the noise is complete. However, this technique will lead to obvious artifacts in reconstructed images when projection data is incomplete. Several attempts have been carried out to settle such problem and overcome its shortcomings [4, 5,8,11,15], which are often considered as optimization algorithms.

However, the problem (2) is an ill-posed problem and its solution will be instable when the projection data is incomplete. In order to surmount the instability of an ill-posed problem, numerous regularization methods have been researched. The thought of compressed sensing (CS) has been widely applied in CT image processing [16-19]. Typically, motivated by the advantage of the total variation (TV) in image denoising [20], TV based image reconstruction model as shown in (3) was proposed to solve few-view and limited-angle problems [8].

$$\min_{x} \| (\nabla x) \|_{1}, s.t. \ Ax = b, x \ge 0.$$
(3)

where the discrete  $(\nabla x)_{i,j} = (x_{i,j} - x_{i-1,j}, x_{i,j} - x_{i,j-1})$ and  $x_{i,j}$  is the intensity value at the position (i, j). Subsequently, adaptive steepest descent projection onto convex sets (ASD-POCS) was proposed to obtain relatively good image with suppressing streak artifacts [9], but it did not work for slope artifacts well. Derivative algorithms of TV [12-14] were put forward to further improve the slope artifacts for limited-angle reconstruction, however, the edge of an object was more or less contortive. Recently,  $l_p(0$  regularization based image reconstruction optimization models are springing up [19,21,22], for example,  $l_p(0 based TV image reconstruction model$ was developed [19] as follows:

$$\min_{x \ge 0} \{ \| (\nabla x) \|_{p}^{p} + \lambda \| Ax - b \|_{2}^{2} \}.$$
(4)

where  $||x||_p = (\sum_i |x_i|^p)^{1/p}$ . There were several methods for solving l<sub>p</sub> based optimization problems such as the reweighted l<sub>1</sub> norm method [23] and the thresholding method [24-27]. Although these TV based algorithms are successful in a large number of situations, the power of them are still limited for preserving gradually varied edges.

In order to preserve the detailed information of an image, some other forms of sparsifying transforms have been developed, such as wavelet frames [28-31], Haar transform [32], S-transform [33], etc.. The core of wavelet frames is on account of the sparsity of some features in an object to be reconstructed. Sparse transform takes some prior knowledge of the object to be reconstructed into account. To measure the transformed result is the second important aspect. As it is known that  $l_0$  based prior can obtain a sparser representation than  $l_1$  based prior [20]. However,  $l_0$ based image reconstruction is often an NP-hard (nondeterministic polynomial-time hard) problem and the objective function will be non-convex and noncontinuous [34]. In the reference [24], the authors proposed an l<sub>1/2</sub> regularizer which has many promising properties such as unbiasedness, sparsity and oracle properties and it can be considered as a representative of  $l_p$  (0 l\_{1/2} based optimization problem is a non-convex problem, and most algorithms for solving that can only provide an approximate local minimizer [35]. Some authors investigated the existence of non-smooth and nonconvex optimization problems [36], and the other authors introduced a proximal alternating linearized minimization (PALM) algorithm to solve such problems [37], where the sequence generated by PALM can be convergent to a stationary point under some conditions.

In this study, we mainly concentrate on  $l_{1/2}$  regularization for wavelet frames based optimization problem to settle the few-view reconstruction within short-scan. To settle this problem, we utilize an alternate direction method (ADM) [37,39,42] to separate the original problem into two subproblems. In the first problem, considering the storage of the system matrix A, we utilize the SART [6] to obtain a proximal point and then use the projection and contraction (PC) [43,44] algorithm to settle the proximal problem. Ultimately, we use the half thresholding (HT) [27] algorithm to settle the second subproblem.

The main contents in this paper are shown below:

1. We use the  $l_{1/2}$  quasi-norm of the wavelet coefficients for the image to develop the image optimization model. After that we give the existence of a solution for the presented model.

2. We utilize the ADM to separate the  $l_{1/2}$  quasinorm regularization term and  $l_2$  norm fidelity term into two subproblems. Then through the idea of the proximal mapping, we utilize SART and PC algorithm to solve the first subproblem, and then use HT algorithm to solve the second subproblem. In addition, we analyze the convergence of special situation of our method for the presented model.

3. To assess the effectiveness of our method, simulated and real experiments are implemented.

This paper is organized as follows. In section 2, we present the corresponding definitions and the fundamental model and then give the existence of the model's solution, and finally we give solving process of our method. In section 3, we represent the convergence analysis of our method, and the simulated and real experiments are given to demonstrate the effectiveness and accuracy of our method. Finally, the conclusions and perspectives are presented in section 4.

# 2 Existence of a Solution

Whether an optimization model has a global solution or not is a key issue. In this section, we first give some definitions and then develop an  $l_{1/2}$  regularization for wavelet frames based image reconstruction model for the few-view CT reconstruction problem within shortscan. In addition, we give the existence of a solution for the presented model.

#### 2.1 Definitions

Some definitions [40] are prepared for the existence of the model's solution in this paper.

The function  $G: \mathbb{R}^m \to \mathbb{R} \bigcup \{\pm \infty\}$  is lower semicontinuous (lsc) at x if  $G(x) = \liminf_{y \to x} G(y)$ , and lower

semi-continuous on  $\mathbb{R}^m$  if this holds for every  $x \in \mathbb{R}^m$ . The level set of G from  $\mathbb{R}^m \to \mathbb{R}$  can be defined as:

$$lev(G,a) \coloneqq \{x \in \mathbb{R}^m \mid G(x) \le a\}.$$
(5)

The asymptotic function  $G_{\infty}$  for any proper function  $G: \mathbb{R}^m \to \mathbb{R} \bigcup \{+\infty\}$  can given by

$$G_{\infty}(y) \coloneqq \inf_{y_n \to y, t_n \to +\infty} \liminf_{n \to +\infty} \frac{1}{t_n} G(t_n y_n).$$
(6)

The kernel of  $G_{\infty}$  can be given as:

$$\ker G_{\infty} := \{ x \in \mathbb{R}^m \mid G_{\infty}(x) = 0 \}.$$
<sup>(7)</sup>

Let  $G: \mathbb{R}^m \to \mathbb{R} \cup \{\pm \infty\}$  be an lsc and proper function. Then G is said asymptotically level stable (als) if for each  $\rho > 0$ , each bounded sequence of reals  $\{\lambda_n\}$ , and each sequence  $\{x_n\} \in \mathbb{R}^n$  satisfying

$$x_n \in lev(G, \lambda_n), \|x_n\|_2 \to +\infty, \frac{x_n}{\|x_n\|_2} \to x^* \in \ker G_\infty$$

there exists  $n_0$  such that  $x_n$ - $\rho x * \in lev(G, \lambda n), \forall n \ge n_0$ .

 $G_{\beta}$  is coercive if  $G_{\beta}(x) \to +\infty$  as  $||x_n||_2 \to +\infty$ , for each  $\beta > 0$ .

#### 2.2 Model

Before presenting our model in this paper, we consider the following general problem:

$$\min_{x \in X} G(x). \tag{8}$$

where G denotes a real and extended functional on a real space X. It is unnecessary that G is convex or smooth. The problem (8) has several contributions in preferences [36,38]. The authors considered that the existence of a solution to problem (8) can be obtained by utilizing a sequence of the problems in which  $\beta > 0$  and it tends to 0:

$$\min_{x \in X} \{ G_{\beta}(x) := G(x) + \frac{\beta}{2} \|x\|_{2}^{2} \}.$$
(9)

The purpose of the term  $\frac{\beta}{2} \|x\|_2^2$  is to guarantee the vistence of a solution to the problem (8) whether G is

existence of a solution to the problem (8) whether G is coercive or not.

In this paper, we mainly concentrate on the image optimization problem. Especially, we put emphasis on the  $l_{1/2}$  quasi-norm of the wavelet frame coefficients since the piecewise smooth function (such as an image) can be sparsely represented using tight wavelet frame transform. That is  $G(x) := \frac{1}{2} ||Ax - b||_D^2 + \lambda ||Wx||_{1/2}^{1/2}$  and  $X = \Omega = \{x | x = (x_1, x_2, \ldots, x_m)^T \in \mathbb{R}^m, x_i \ge 0, i = 1, 2, \ldots, m\}$  which can be counted as  $R_+^m$ . Then, the presented model can be expressed as

$$\min_{x\in\Omega} \{ \frac{1}{2} \| Ax - b \|_{D}^{2} + \lambda \| Wx \|_{1/2}^{1/2} \}.$$
 (10)

where x is the image vector comprised of pixel coefficients;  $A: \Omega \rightarrow R_{+}^{l}$  is a bounded linear operator;  $b \in R_{+}^{l}$  is a projection vector; D denotes the positive definite diagonal matrix; W :  $\mathbb{R}^{m} \rightarrow \mathbb{R}^{q}$  is a multilevel wavelet tight framelets transform operator;  $\lambda$  is a parameter balancing the fidelity term and regularization term;  $\|\cdot\|_{1/2}$  denotes the  $l_{1/2}$  quasi-norm, i.e.,  $\|x\|_{1/2} = (\sum_{i=1}^{m} |x_{i}|^{1/2})^{2}$ . The piece constant linear B-spline framelets transform W which can be reconstructed by reference [31] is utilized in this paper.

#### 2.3 Existence

A family of problems  $G_{\beta} (\beta \rightarrow 0^+)$  to approximate the problem (10) are adopted to accomplish the existence of a solution. First, we need a corollary to accomplish the existence of a solution for the presented model.

**Lemma 2.1.** (Corollary 3.4.2 of [40]) Let G:  $\Omega \subseteq \mathbb{R}^m \rightarrow [0, +\infty]$  be a function and  $\{\gamma_n\}$  be a real bounded

sequence satisfying: (i)  $G_{\beta}$  is coercive;

(ii) G is als,  $\frac{x_n}{\|x_n\|_2} \to x^*(as \ n \to +\infty)$  strongly in  $\Omega$ ,

where  $x_n \in \text{lev}(G, \gamma_n)$  and  $||x_n||_2 \to +\infty$ .

Then, the problem (8) admits a global solution.

**Remark 1**. Lemma 2.1 implies that there exists a subsequence extracted from  $G_{\beta}$  ( $\beta \rightarrow 0^+$ ) which is convergent to a global solution.

To prove the existence of (10), we should validate the conditions of Lemma 2.1 and consider the problem as follows:

$$\underset{x \in \Omega}{\operatorname{argmin}} \{ \frac{1}{2} \| Ax - b \|_{D}^{2} + \lambda \| Wx \|_{1/2}^{1/2} + \frac{\beta}{2} \| x \|_{2}^{2} \} (\beta \to 0^{+}). (11)$$

For simplicity, we let

$$G(x) := \frac{1}{2} \left\| Ax - b \right\|_{D}^{2} + \lambda \left\| Wx \right\|_{1/2}^{1/2},$$
(12)

$$G_{\beta}(x) := G(x) + \frac{\beta}{2} \|x\|_{2}^{2}.$$
 (13)

**Theorem 2.2.** (The existence of the presented model): Let G and  $G_{\beta}$  be defined as (12) and (13), respectively.  $A: \Omega \subseteq \mathbb{R}^m \to \mathbb{R}^l_+$  is a bounded linear operator. W: $\mathbb{R}^m \to \mathbb{R}^q$  is a bounded linear operator and W<sup>T</sup>W = I. Then, the problem (10) admits a global solution.

**Proof.** First of all, it is clear that  $G_{\beta}$  is coercive since  $G_{\beta}(x) \to +\infty$  as  $||x_n||_{\gamma} \to +\infty$  for each  $\beta > 0$ .

Secondly, consider  $y^n, y^* \in \Omega \subseteq \mathbb{R}^m$  and  $y^n \to y^*$  (as  $n \to +\infty$ ). Since W is a bounded linear operator, it follows that  $Wy^n \to Wy^*$  (as  $n \to +\infty$ ). Thus,  $\|Wy_n\|_{1/2}^{1/2} \to \|Wy^*\|_{1/2}^{1/2}$  (as  $n \to +\infty$ ). This implies that the  $l_{1/2}$  term of G is lsc. Thus G is a proper and lsc function.

In addition, since  $||x_n||_2 \to +\infty$  and  $x_n \in \text{lev}(G, \lambda_n)$ , G(x<sub>n</sub>) is bounded. According to the definition of G<sub>∞</sub>(x\*), let  $y_n = \frac{x_n}{||x_n||_2}$  and  $t_n = ||x_n||_2$ . It immediately

obtains that  $G_{\infty}(x^*) \ge 0$  and

$$G_{\infty}(x^{*}) \leq \liminf_{n \to +\infty} \frac{1}{\|x_{n}\|_{2}} G(\|x_{n}\|_{2} \cdot \frac{x_{n}}{\|x_{n}\|_{2}}) = \liminf_{n \to +\infty} \frac{G(x_{n})}{\|x_{n}\|_{2}} = 0.$$
  
Thus G. (x\*) and x\*  $\in \ker G$ .

Thus  $G_{\infty}(x^*)$  and  $x^* \in \ker G_{\infty}$ .

Furthermore, since 
$$\frac{x_k}{\|x_k\|_2} \to x^*$$
 (as  $k \to +\infty$ ), it

follows that 
$$\left\| W(x_n - \rho \cdot \frac{x_k}{\|x_k\|_2}) \right\|_{1/2}^{1/2} \to \left\| W(x_n - \rho \cdot x^*) \right\|_{1/2}^{1/2}$$
.

Let n = k and  $\rho \in [0, ||x_k||_2]$ , we have

$$\left\| W(x_n - \rho \cdot \frac{x_k}{\|x_k\|_2}) \right\|_{1/2}^{1/2} = \left\| (1 - \frac{\rho}{\|x_k\|_2}) \cdot Wx_k \right\|_{1/2}^{1/2} \le \left\| Wx_k \right\|_{1/2}^{1/2}.$$

For  $\forall \varepsilon > 0$ , there exists  $n_0 > 0$ , when  $n = k > n_0$ , it obtains that

$$\left\|W(x_n - \rho x^*)\right\|_{1/2}^{1/2} = \left\|W(x_n - \rho \cdot \frac{x_k}{\|x_k\|_2})\right\|_{1/2}^{1/2} + \varepsilon \le \left\|Wx_k\right\|_{1/2}^{1/2} + \varepsilon.$$

That is  $G(x_n, \rho x^*) \leq G(x_n)$ , for  $\rho \in [0, ||x_n||_2]$  and

 $n > n_0$ . And thus  $x_n - \rho x^* \in lev(G, \lambda_n)$ .

According to lemma 2.1, the problem (10) admits a global solution.

#### 2.4 Solving Process

According to remark 1, we can solve the problem (11) to obtain the solution of the problem(10). In this section, numerical implementation of the problem (11) is presented. The alternating direction method (ADM) is utilized to deal with this problem. Considering the storage of the system matrix A, we will incorporate the SART with the ADM, and then use projection contraction (PC) algorithm to solve the constraint problem. For the  $l_{1/2}$  quasi-norm, we will use the half thresholding (HT) algorithm. we let (xi,j) denote a discretized image on a rectangular grid  $m_1 \times m_2$ , i = 1, 2, . . . ,  $m_1$ ,  $j = 1, 2, \ldots$  ,  $m_2$ , and  $(b_{i,j})$  denote a discretized image space of the function A on a rectangular grid  $l_1 \times l_2$ ,  $i = 1, 2, ..., l_1, j = 1, 2, ..., l_2$ . Where  $l_1$  is the number of the detector bins and  $l_2$  is the number of the projection angle. The projection function A can be defined by the basis functions which are given by the seventh formula in reference [7]. Then, (x<sub>i,j</sub>) and (b<sub>i,j</sub>) are rearranged into column vectors  $x \in R^m_+$  and  $b \in R^l_+$ , respectively. The piecewise constant B-spline framelets can be reconstructed by reference [31], and the associated fifilters are as follows

$$h_0 = \frac{1}{4}[1,2,1], h_1 = \frac{\sqrt{2}}{4}[-1,0,1], h_2 = \frac{1}{4}[-1,2,-1].$$

The key to dealing with the presented model is to de-couple the D-weighted  $l_2$  and  $l_{1/2}$  portions of the model. It is difficult to deal with the problem directly. Fortunately, ADM [37] is widely adopted to separate the original problem (11) into two sub-problems. The problem (11) can be converted into:

$$\underset{x \in \Omega, \alpha}{\operatorname{argmin}} \{ \frac{1}{2} \| Ax - b \|_{D}^{2} + \lambda \| \alpha \|_{1/2}^{1/2} + \frac{\beta}{2} \| x \|_{2}^{2} \}, \, s.t. \, \alpha = Wx.$$
 (14)

Using the augmented lagrangian method, it implies that

$$\underset{x \in \Omega, \alpha, u}{\operatorname{argmin}} \{ \frac{1}{2} \| Ax - b \|_{D}^{2} + \lambda \| \alpha \|_{1/2}^{1/2} + \frac{\gamma}{2} \| Wx - \alpha + u \|_{2}^{2} + \frac{\beta}{2} \| x \|_{2}^{2} \}. (15)$$

The ADM minimizes the problem (15) by iteratively minimizing x and  $\alpha$  alternately. Its (n + 1)th process can be expressed as follows:

step1: 
$$\arg\min_{x\in\Omega} \{\frac{1}{2} \|Ax - b\|_{D}^{2} + \frac{\gamma}{2} \|Wx - \alpha^{n} + u^{n}\|_{2}^{2} + \frac{\beta}{2} \|x\|_{2}^{2}\},$$
 (16)

step 2: 
$$\arg\min_{x\in\Omega} \{\lambda \|\alpha\|_{1/2}^{1/2} + \frac{\gamma}{2} \|Wx^{n+1} - \alpha + u^n\|_2^2\},$$
 (17)

step3: 
$$u^{n+1} = u^n - (\alpha^{n+1} - Wx^{n+1}).$$
 (18)

For convenience, we denote:

$$f(x) := \frac{1}{2} \|Ax - b\|_{D}^{2}, h(x, \alpha, u) := \frac{\gamma}{2} \|Wx - \alpha + u\|_{2}^{2};$$
(19)

$$v(\alpha) := \lambda \|\alpha\|_{1/2}^{1/2}, g_{\gamma,\beta}(x,\alpha,u) := \frac{\gamma}{2} \|Wx - \alpha + u\|_{2}^{2} + \frac{\beta}{2} \|x\|_{2}^{2}.$$
(20)

Although the problem (16) has a closed form solution, the system matrix A is difficult to store in memory considering the practical CT projection. To solve the step 1, in this paper, we utilize SART to get a proximal point by minimizing the objective function  $\frac{1}{2} \|Ax - b\|_D^2$ , and the proximal point can be expressed as:

$$\hat{x}^n \coloneqq x^n - \frac{\omega}{\tau} V^{-1} \nabla f(x^n), \qquad (21)$$

where  $\omega/\tau$  denotes a relaxation parameter in (0,2); *V* is the diagonal matrix with diagonal non-zero element  $\sum_{i=1}^{l} a_{i,j}, j = 1, 2, ..., m$ ;  $V^{-1}$  is the inverse transform of *V*. *D* is the inverse matrix of the diagonal matrix with non-zero diagonal element  $\sum_{j=1}^{m} a_{i,j}, i = 1, 2, ..., l$ . Then, the problem (16) can be converted into the following form:

$$x^{n+1} \in \underset{x \in \Omega}{\operatorname{arg\,min}} \{ \frac{\tau}{2} \| x - \hat{x}^n \|_2^2 + g_{\gamma,\beta}(x, \alpha^n, u^n) \}, \quad (22)$$

where  $\tau$  is a balancing parameter. Note that there are three parameters to be tuned in (22). In fact, (22) expresses the proximal forward-backward scheme [37]. Using the proximal mapping, it follows that

$$\Pr ox_{\tau}^{g_{\gamma,\beta}}(\hat{x}^{n}) := \arg\min_{x\in\Omega} \{\frac{\tau}{2} \|x - \hat{x}^{n}\|_{2}^{2} + g_{\gamma,\beta}(x,\alpha^{n},u^{n})\}.$$
(23)

Then it can obtain

$$x^{n+1} \in \Pr{ox_{\tau}^{g_{\gamma,\beta}}(\hat{x}^n)}.$$
 (24)

Afterwards, the PC algorithm [43, 44] is utilized to solve (24). The solution process of (16) includes two steps as follows:

1. Obtain a proximal point  $\hat{x}^n$  using the SART;

2. Optimize (24) using PC algorithm to obtain  $x^{n+1}$ . That can be expressed as  $x^{n+1} = PC_{\Omega}(\hat{x}^n)$ .

Denote 
$$F(x) := \frac{\tau}{2} \|x - \hat{x}^n\|_2^2 + g_{\gamma,\beta}(x, \alpha^n, u^n).$$

The implementation process of PC algorithm [43,44] is as following:

Table 1. The pseudo-code of PC algorithm

Implementation steps of the PC algorithm  
Step 1. Initialize 
$$t_0 = 1, v \in (0,1), \rho = 1.9, x^0 \in \Omega$$
 and  $k = 0$   
Step 2.  $\widetilde{x}^k = PC_{\Omega}[x^k - t_k \nabla F(x^k)],$   
 $r_k = \frac{t_k \|\nabla F(x^k) - \nabla F(\widetilde{x}^k)\|_2}{\|x^k - \widetilde{x}^k\|_2},$ 

While 
$$r_k > \upsilon, t_k := \frac{2}{3}t_k \cdot \min\{1, \frac{1}{r_k}\},$$
  
 $\widetilde{x}^k = PC_{\Omega}[x^k - \beta_k \nabla F(x^k)],$   
 $r_k = \frac{t_k \left\| \nabla F(x^k) - \nabla F(\widetilde{x}^k) \right\|_2}{\left\| x^k - \widetilde{x}^k \right\|_2},$   
end (While)  
 $d(x^k, \widetilde{x}^k) = (x^k - \widetilde{x}^k) - t_k [\nabla F(x^k) - \nabla F(\widetilde{x}^k)],$   
 $e_k = \frac{(x^k - \widetilde{x}^k)^T d(x^k, \widetilde{x}^k)}{\left\| d(x^k, \widetilde{x}^k) \right\|_2^2},$   
 $x^{k+1} = x^k - \rho e_k d(x^k, \widetilde{x}^k).$   
If  $r_k < 0.4$  then  $t_k := 1.5t_k$ , end (if)  
Step 3.  $t_{k+1} := t_k$  and  $k := k+1$ , go back to Step 2.

To solve the problem (17), we rewrite it as

$$\alpha^{n+1} \in \arg\min_{\alpha} \{\lambda \|\alpha\|_{1/2}^{1/2} + \frac{\gamma}{2} \|Wx^{n+1} - \alpha + u^n\|_2^2\}.$$
 (25)

The solution of the problem (25) has a closed form and it can be effectively settled using the half thresholding (HT) algorithm [27] as follows.

$$\alpha_i^{n+1} = HT_{\tilde{\lambda}}(\alpha_i^n) = \begin{cases} 0 & \text{if } |\alpha_i^n| \le p(\lambda), \quad (26) \\ \frac{2}{3}a_i^n(1 + \cos(\frac{2\pi}{3} - \frac{2}{3}\varphi_{\tilde{\lambda}}(a_i^n))) & \text{otherwise.} \end{cases}$$
  
Where  $a_i^n = (Wx^{n+1})_i + u_i^n, \tilde{\lambda} = \frac{2\lambda}{\gamma}, p(\tilde{\lambda}) = \frac{\sqrt[3]{54}}{4}(\tilde{\lambda})^{2/3}$   
and  $\varphi_{\tilde{\lambda}}(a) = \arccos(\frac{\tilde{\lambda}}{8}(\frac{|a|}{3})^{-3/2}).$ 

The step 3 is to update the lagrangian multiplier.

The above process including three steps can be regarded as SART-PC-HT method (our method). We denote Nmax the maximum number of the iteration, and is the relative error between  $x^{n+1}$  and  $x^n$ . The pseudo-code of SART-PC-HT method can be described as shown in Table 2.

Table 2. The pseudo-code of SART-PC-HT method

| Implementation steps of the SART-PC-HT reconstruction   |
|---|
| method  |
| Step 1. Initialize:   |
| Given $\varepsilon_0, \lambda, \gamma > 0, \beta = 1, x^1 = 0, \alpha^1 = W x^1, u^1 = \alpha^1, t = 1$ |
| $\omega=\tau=n=1.$  |
| While ( $\iota > \mathcal{E}_0$ and $n < N_{\max}$ )  |
| Step 2. SART-PC update:   |
| (1) SART update: $\hat{x}^n = x^n - \omega V^{-1} \nabla f(x^n)$  |
| (2) PC update: $x^{n+1} = PC_{\Omega}(\hat{x}^n)$   |
| Step 3. HT update:  |
| $\alpha^{n+1} = HT_{\tilde{\lambda}}(Wx^{n+1} + u^n)$   |
| Step 4. Lagrangian Multiplier update:   |
| $u^{n+1} \coloneqq u^n - (\alpha^{n+1} - Wx^{n+1})$   |

$$\iota = \frac{\left\| x^{n+1} - x^n \right\|_2}{\left\| x^{n+1} \right\|_2}$$
$$\beta_{n+1} \coloneqq 0.9 * \beta_n, n \coloneqq n+1$$
end (While)  
Output  $x^{n+1}$ 

## **3 Simulated and Real Experiments**

In this section, simulated and real experiments are implemented to verify the effectiveness of our method. We utilize the simulated experiments with ten different levels of Guassian noise to do the quantitative statistics interpretations which include RMSE and PSNR [41] as follows:

$$RMSE = \sqrt{\frac{1}{m} \sum_{i=1}^{m} (x(i) - x_r(i))^2},$$
 (27)

$$PSNR = 10\log_{10} \frac{\|x_r\|_{\infty}^2}{RMSE^2}.$$
 (28)

where x denotes the reconstructed image,  $x_r$  is the reference image and m is the total number of the image pixels.

Then, we use the real experiments to conform the effectiveness of our method, compared with the commercial FBP method [6]. In the experiments, the parameters of our method are chose by trial and error for the best reconstructed image quality. All the experiments are implemented on 2.90 GHz intel(R) Pentium(R) G2020 CPU processor with 4G memory and coded.

#### 3.1 Simulated and Real Experiments

We test and verify the effectiveness of our method for few-view reconstruction within short-scan using a digital NURBS based cardiac torso (NCAT) phantom [45]. The simulated projection data are generated by projecting a 256×256 discretized NCAT phantom with adding ten different levels of Gaussian noise, where all the mean value of Gaussian noise are zero, and their standard deviations are 0.1%, 0.2%,..., 1% of the current projection values (cpv), respectively. The geometry scanning parameters for CT imaging system are listed in Table 3. The number of projection views, the parameters of our method and the corresponding statistical analysis RMSE and PSNR are listed in Table 2. The stopping criterion is reaching the maximum iteration number  $N_{\text{max}} = 100$  for ten different levels of Gaussian noise. In PC algorithm, maximum iteration number is 2000 and stopping criterion is  $\varepsilon = 1 \times 10^{-5}$ , and adding a stopping criterion is  $\varepsilon_0 = 1 \times 10^{-6}$  to our method.

As shown in Table 4, with the decrease of the standard deviation and the increase of the projection views, RMSE is decreasing and PSNR is increasing. Meanwhile, the main parameters of our method are

chose differently with diverse standard deviation.

Figure 2 demonstrates some of the reconstructed results from ten different levels of Gaussian noise. As shown in Figure 1, (a) is an original image for NCAT, and the rest of them are reconstructed using our method from different projection views by adding Gaussian noise with standard deviation 0.3%cpv, 0.5%cpv, 0.8%cpv and 1%cpv, respectively. From the enlarged images in Figure 2, with the decrease of the standard deviation and the increase of the projection views, the noises are restrained effectively and details are maintained well.

With the analysis of Figure 3 and Figure 4, the RMSE of the reconstructed NCAT phantoms is gradually decreasing and their PSNR are increasing with the standard deviation decreasing and the projection views increasing. That is, the smaller standard deviation and more complete projection views will lead the sequence  $\{x_n\}$  to converge to a stationary point which is very close to a global point. Meanwhile, the bigger standard deviation and fewer projection views will make the sequence  $\{x_n\}$  converge to a stationary point early, but the reconstructed result is not good for the details.

 
 Table 3. Geometry scanning parameters of NCAT for simulated CT imaging system

| The distance from X-ray source to rotation center               | 900.0mm                 |
|---|-------------------------|
| The distance from detector to rotation center                   | 400.0mm                 |
| The angle between two adjacent projection views(interval angle) | $0.6667^{0}$            |
| The angle between two adjacent rays                             | 0.00110                 |
| The number of detector units                                    | 256                     |
| The diameter of field of view                                   | 255.0mm                 |
| Pixel size  | 1.0 ×1.0mm <sup>2</sup> |
| Image size  | 256 × 256               |

 
 Table 4. Quantitatively assess the image reconstruction quality of NCAT in the 100th iteration

| Projection<br>views | Standard deviation | λ       | γ      | RMSE   | PSNR    |
|---------------------|--------------------|---------|--------|--------|---------|
|                     | 0.3%cpv            | 0.00063 | 0.0090 | 0.0090 | 40.9608 |
| 272p-data           | 0.5%cpv            | 0.0009  | 0.0120 | 0.0128 | 37.8307 |
|                     | 0.8%cpv            | 0.0045  | 0.0220 | 0.0225 | 32.9697 |
|                     | 1%cpv              | 0.0070  | 0.0280 | 0.0294 | 30.6218 |
| 92p-data            | 0.3%cpv            | 0.00063 | 0.0090 | 0.0109 | 39.2420 |
|                     | 0.5%cpv            | 0.0009  | 0.0120 | 0.0148 | 36.5669 |
|                     | 0.8%cpv            | 0.0045  | 0.0220 | 0.0266 | 31.5029 |
|                     | 1%cpv              | 0.0070  | 0.0280 | 0.0345 | 29.2497 |
| 56p-data            | 0.3%cpv            | 0.00063 | 0.0090 | 0.0132 | 32.9580 |
|                     | 0.5%cpv            | 0.0009  | 0.0120 | 0.0169 | 32.2713 |
|                     | 0.8%cpv            | 0.0045  | 0.0220 | 0.0318 | 29.8762 |
|                     | 1%cpv              | 0.0070  | 0.0280 | 0.0401 | 28.6028 |



**Figure 2.** (a) is a reference image for NCAT. (b1)-(d4) are reconstructed images using our method from 272, 92 and 56 projection views (denote by 272p-data, 92p-data and 56p-data) by adding Gaussian noise with standard deviation 0.3%cpv, 0.5%cpv, 0.8%cpv and 1%cpv in 100th iteration, respectively. The lower left corner and upper right corner images are the enlarged images of the red rectangle boxes. The angular scope is  $[0, 181^{\circ}]$ . The display window is [0, 1] cm<sup>-1</sup>.



**Figure 3.** The RMSE surfaces for NCAT from upper layer to lower layer demonstrate the reconstructed images from 56 projection views by adding Gaussian noise with standard deviation 0.1%cpv, 0.2%cpv,...,1%cpv, respectively.



**Figure 4.** The PSNR surfaces for NCAT from lower layer to upper layer demonstrate the reconstructed images from 56 projection views by adding Gaussian noise with standard deviation 0.1%cpv, 0.2%cpv,...,1%cpv, respectively.

#### 3.2 Reconstruction from Real Walnut Data

To further confirm the effectiveness and accuracy of our method for few-view reconstruction within shortscan, we use the real Walnut projection data which can be well known in reference [46]. Some details of the Walnut are difficult to reconstruct since its projection data contains some noises. In order to highlight our method's effectiveness to few-view reconstruction, we adopt the different projection views in [0,180°], compared with commercial FBP method.

The size of Walnut image to be reconstructed is  $656 \times 656$ . The real CT imaging system can refer to [46]. In the experiments, the stopping criterion is reaching  $N_{max} = 100$ , and the main parameters of our method in this subsection are  $\lambda = 0.00003$  and  $\gamma = 0.003$ . In SART, the parameter  $\omega = 1$ . In PC algorithm,  $\tau = 1$ , maximum iteration number is 2000 and stopping criterion is  $\varepsilon = 1 \times 10^{-5}$ , and the other stopping criterion is  $\varepsilon_0 = 1 \times 10^{-6}$  to our method.

Figure5 and Figure 6 demonstrates the reconstructed Walnut images using commercial FBP method and our method. As shown in Figure 5, the first row are the images using commercial FBP method from 600, 300 and 200 projection views in angular scope [0, 180°], respectively; the third row are the results using our method from 600, 300 and 200 projection views in angular scope [0, 180°], respectively; the second row and third row are the enlarged images of the red rectangle box of the first row and third row, respectively; the lower left corner images of the first and third row are the enlarged images of the red rectangle box. As shown in Figure 6, the reconstructed Walnut images using commercial FBP method and our method are reconstructed from 150, 100 and 76 projection views in angular [0, 180°], respectively. The layout of the rest in Figure 6 is as shown in Figure 5.



**Figure 5.** In the first row, the three images using commercial FBP method are reconstructed from 600, 300 and 200 projection views in angular scope  $[0, 180^{\circ}]$ , respectively. The images of the third row using our method are the 100th iterative results reconstructed from 600, 300 and 200 projection views in angular scope  $[0, 180^{\circ}]$ , respectively. The second and forth row are the corresponding enlarged images of the red rectangle box of the first and third row. The lower left corner images of the first and third row are the enlarged images of the red rectangle box. The red arrows indicate the fuzzy or broken parts. The display window is [0.15, 0.85]cm<sup>-1</sup>.



**Figure 6.** In the first row, the three images using commercial FBP method are reconstructed from 150, 100 and 76 projection views in angular scope  $[0, 180^{\circ}]$ , respectively. The images of the third row using our method are the *100*th iterative results reconstructed from 150, 100 and 76 projection views in angular scope  $[0, 180^{\circ}]$ , respectively. The second and forth row are the corresponding enlarged images of the red rectangle box of the first and third row. The lower left corner images of the first and third row are the enlarged

images of the red rectangle box. The red arrows indicate the fuzzy or broken parts. The display window is [0.15, 0.85] cm<sup>-1</sup>

According to Figure 5 and Figure 6, it is observed that the Walnut images using FBP method contain the larger noise and some details of them are distorted along with the decrease of the projection views. Although the number of the projection views are decreasing, the results using our method have relatively high quality and contrast, except the rupture of the filament at the top of the Walnut image. It can be seen our method outperforms the commercial FBP method in suppressing noise and preserving details, and our method has the ability for reducing the projection views to lower the radiation exposure while maintaining relatively high image quality.

## **4** Conclusions and Perspectives

To deal with the few-view reconstruction problem within shortscan, we adopt  $l_{1/2}$  regularization for wavelet frames based image reconstruction model, and adopt the SART-PC-HT method to solve it. First, the existence of the presented model is analyzed through the existing corollary. Second, ADM is utilized to separate the original problem into two subproblems, which can be settled using SART, PC algorithm and HT algorithm, respectively. Third, simulated NCAT experiment and real Walnut experiment confirm that our method can obtain the reconstructed images with relatively good quality and high contrast while decreasing the projection views. That means our method has the ability to settle the few-view reconstruction problem within short-scan whose projection data contains noise. However, when the projection data are very incomplete and has high-level noise, the details of the reconstructed images using our method will not be preserved very well.

In our method, the parameters are chose by trial and error, which are not adaptive for any given projection data. In the future, we will analyze the adaption of the parameters in our method. For high level noise and few projection views, our method will be not effective, which will be improved with taking the type of the noise of the projection data into consideration.

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### References

[1] T. M. Buzug, "Computed tomography: from photon

statistics to modern cone-beam CT" 1nd edition, Springer-Verlag, Berlin Heidelberg, 2008.

- [2] D. J. Brenner and E. J. Hall, Computed tomography-an increasing source of radiation exposure, *New England Journal of Medicine*, 357 (2007), 2277-2284.
- [3] A. Berrington de Gonzalez, M. Mahesh and K.-P. Kim, Projected cancer risks from computed tomographic scans performed in the United States in 2007, *Journal of Vascular Surgery*, 51(3) (2009), 2071-2077.
- [4] Jiang Hsieh, Adaptive streak artifact reduction in computed tomography resulting from excessive X-ray photon noise, *Medical Physics*, 25(11) (2009), 2139-2147.
- [5] T. F. Li, X. Li, J. Wang, J. H. Wen, H. B. Lu and Z. R. Liang, Nonlinear sinogram smoothing for lowdose X-ray CT, *IEEE Transactions on Nuclear Science*, 51(5) (2004), 2505-2513.
- [6] A. C. Kak and M. Slaney, "Principles of computerized tomographic imaging" New York: IEEE Press, 1987.
- [7] A. H. Andersen and A. C. Kak, Simultaneous Algebraic Reconstruction Technique (SART): A Superior Implementation of the Art Algorithm, *Ultrasonic Imaging*, 6(1) (1984), 81-94.
- [8] E. Y. Sidky, C. M. Kao and X. Pan, Accurate image reconstruction from few-views and limited-angle data in divergent-beam CT, arXiv: 0904.4495.
- [9] E. Y. Sidky and X. Pan, Image reconstruction in circular cone-beam computed tomography by constrained, total-variation minimization, *Physics in Medicine and Biology*, **53**(17) (2008), 4777-4807.
- [10] G. Herman and R. Davidi, On Image Reconstruction from a Small Number of Projections, *Inverse Problems*, 24(4) (2008), 45011C45028.
- [11] H. Yu and G. Wang, A soft-threshold fifiltering approach for reconstruction from a limited number of projections, *Physics in Medicine and Biology*, **55**(13) (2010), 3905-3916.
- [12] L. Ritschl, F. Bergner, C. Fleischmann and M. Kachelriess, Improved total variation-based CT image reconstruction applied to clinical data, *Physics in Medicine and Biology*, 56(6) (2011), 1545-1561.
- [13] Z. Chen, X. Jin, L. Li and G. Wang, A limitedangle CT reconstruction method based on anisotropic TV minimization, *Physics in Medicine* and Biology, 58(7) (2013), 2119-2141.
- [14] X. Lu, Y. Sun and Y. Yuan, Optimization for limited angle tomography in medical image processing, *Pattern Recognition*, 44 (2011), 2427C2435.
- [15] Q. Xu, H. Y. Yu, X. Q. Mou, L. Zhang, J. Hsieh and G. Wang, Low-dose X-ray CT Reconstruction via Dictionary Learning, IEEE

Trans. Medical Imaging, 31(9) (2012), 1682-1697.

- [16] D. L. Donoho, Compressed sensing, *IEEE Transactions on Information Theory*, **52** (2006), 1289-1306.
- [17] Y. J. Lu, X. Q. Zhang, A. Douraghy, D. Stout, J. Tian, T. F. Chan and A. F. Chatziioannou, Source Reconstruction for Spectrally-resolved Bioluminescence Tomography with Sparse A priori Information, *Optics Express*, **17** (2009), 8062-8080.
- [18] X. Mou, J. Wu, T. Bai, Q. Xu, H. Yu and G. Wang, Dictionary learning based low-dose x-ray CT reconstruction using a balancing principle, *Proceedings of the SPIE*, **9212** (2014).
- [19] E. Y. Sidky, R. Chartrand, J. M. Boone and X. Pan, Constrained TpV Minimization for Enhanced Exploitation of Gradient Sparsity: Application to CT Image Reconstruction, *IEEE Journal of Translational Engineering in Health* and Medicine, 2 (2014), 1-18.
- [20] Rudin Leonid I., S. Osher and E. Fatemi, Nonlinear total variation based noise removal algorithms, *Physica D: Nonlinear Phenomena*, **60** (1992), 259-268.
- [21] Mianyi C., Deling M., Peng H., et al., A CT Reconstruction Algorithm Based on L1/2 Regularization, Computational and Mathematical Methods in Medicine, 2014 (2014).
- [22] C. Miao and H. Yu, A General-Thresholding Solution for lp (0 ) Regularized CTReconstruction,*IEEE Transactions on Image Processing*,**24**(2015), 5455-5468.
- [23] E. J. Candes, M.B. Wakin and S.P. Boyd, Enhancing Sparsity by Reweighted `1 Minimization, *Journal of Fourier Analysis and Applications*, 14 (2008), 877-905.
- [24] Z. Xu, H. Zhang, Y. Wang, X. Chang and Y. Liang, L<sub>1/2</sub> regularization, *Science China Information Sciences*, **53** (2010), 1159-1169.
- [25] Z. Xu, X. Chang, F. Xu and H. Zhang, L1/2 Regularization: A Thresholding Representation Theory and a Fast Solver, *IEEE Transactions on Neural Networks and Learning Systems*, 23 (2012), 1013-1027.
- [26] D. A. Lorenz, Convergence rates and source conditions for Tikhonov regularization with sparsity constraints, *Journal of Inverse and Illposed Problems*, **16** (2008), 463C478.
- [27] W. Cao, J. Sun and Z. Xu, Fast image deconvolution using closed-form thresholding formulas of  $L_q$  (q = 1/2, 2/3) regularization, *Journal of Visual Communication and Image Representation*, **24** (2013), 31-41.
- [28] X. Jia, B. Dong, Y. Lou and B. Steve B Jiang, GPU-based iterative cone-beam CT reconstruction using tight frame regularization, *Physics in Medicine and Biology*, **56** (2011), 3787-3807.

- [29] H. Gao, J. F. Cai, Z. Shen and H. Zhao, Robust principal component analysis-based fourdimensional computed tomography, *Physics* in Medicine and Biology, 56 (2011), 3181-3198.
- [30] B. Zhao, H. Gao, H. Ding, and S. Molloi, Tightframe based iterative image reconstruction for spectral breast CT, *Medical Physics*, 40 (2013), 031905.
- [31] A. Ron and Z. Shen, Affiffiffine systems in L2(Rd) II: Dual systems, *Journal of Fourier Analysis and Applications*, 3 (1997), 617-637.
- [32] E. Gardu no, Herman, G. T. and R. Davidi, Reconstruction from a Few Projections by l<sub>1</sub>-Minimization of the Haar Transform, *Inverse Problem*, 27 (2011), 55006-55018.
- [33] J. Luo, J. Liu, W. Li, Y. Zhu and R. Jiang, Image Reconstruction from Sparse Projections Using S-Transform, *Journal of Mathematical Imaging and Vision*, 43 (2012), 227-239.
- [34] E. J. Candes, J. Romberg and T. Tao, Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information, *IEEE Transactions on Information Theory*, **52** (2006), 489-509.
- [35] X. Chen, F. Xu and Y. Ye, Lower Bound Theory of Nonzero Entries in Solutions of l<sub>2</sub>-l<sub>p</sub> Minimization, *IEEE Transactions on Information Theory*, **32** (2010), 2832C2852.
- [36] Kazufumi Ito and Karl Kunisch, A Note on the Existence of Nonsmooth Nonconvex Optimization Problems, *Journal of Optimization Theory and Applications*, **163** (2014), 697-706.
- [37] J. Bolte, S. Sabach and M. Teboulle, Proximal alternating linearized minimization for nonconvex and nonsmooth problems, *Mathematical Programming*, **146** (2013), 459-494.
- [38] Baiocchi C, Buttazzo G, Gastaldi F, et al., General existence theorems for unilateral problems in continuum mechanics, *Mathematical Programming*, **100** (1988), 149-189.
- [39] J. F. Cai, S. Osher and Z. Shen, Split Bregman Methods and Frame Based Image Restoration, *Siam Journal on Multiscale Modeling and Simulation*, **8**(2) (2009), 337C369.
- [40]Alfred Auslender and Marc Teboulle, "Asymptotic Cones and Functions in Optimization and Variational Inequalities" Springer Monographs in Mathematics, 2003.
- [41] Storath M, Weinmann A, Frikel J and Unser M, Joint image reconstruction and segmentation using the Potts model, *Inverse Problems*, **31** (2015), 25003-25031.
- [42] Y. Xiao, J. Yang and X. Yuan, Alternating algorithms for total variation image reconstruction from random projections Inverse Problems and Imaging, *Inverse Problems and Imaging*, 6 (2012), 547-563.
- [43] B. He, A class of projection and contraction

methods for monotone variational inequalities, *Applied Mathematics and Optimization*, **35** (1997), 69-76.

- [44] B. HeM. H. Xu, A general framework of contraction methods for monotone variational inequalities, *Pacifific Journal of Optimization*, 4 (2008), 195-212.
- [45] Xcat phantom website Available: http://www.hopkinsradiology.org/DMIP/Research /xcat.
- [46] Tomographic X-ray data of a walnut, http://www.fifips.fifi/dataset.php.