

A series of equivalent area circle yield criteria based on unified strength theory

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Abstract: Unified strength theory considering the influence of the intermediate principal stress is widely used in geotechnical engineering, but the singularities bring inconvenience to the numerical calculation. A series of equivalent area circle yield criteria based on unified strength theory are derived. The parameters of the new yield criteria and Drucker-Prager criteria are discussed, and the flow vector coefficients of the new yield criteria are given. The new series of yield criteria are very convenient for numerical calculation and can be served as reference for the evaluation of the effects of strength theory.

1 Introduction

The strength theory of geotechnical engineering provides the criteria to analyze the yield and failure of geotechnical materials and structures, which also plays an essential role on stability research. At present, hundreds of strength criteria have been proposed, forming a rich theoretical system of strength theory. This theoretical system can be classified from different perspectives. According to the physical quantity of the expression, it can be divided into stress form, strain form and energy form. It can also be divided into linear strength theory and nonlinear strength theory, depending on the property of the yield function. According to the number of parameters, it can be divided into single parameter, double parameters, and multiple parameters. In addition, there are some other classification methods (e.g., according to the characteristics of the limit lines, the characteristics of the off-plane trace, the number of yield surfaces). The representative strength criteria used in geotechnical engineering mainly include: Tresca criterion, Mohr-Coulomb criterion, Mogi-Coulomb criterion, Huber-Mises criterion, Drucker-Prager criterion, Matsuoka-Nakai criterion, Zienkiewicz-Panda criterion, etc^[1]. Among of which, Mohr-Coulomb criterion is the most widely used and most controversial strength theory in geotechnical engineering. The main problem is that it does not consider the intermediate principal stress. A large number of experimental studies have shown that the intermediate principal stress has a significant effect on the yield and failure of the material^[2, 3]. Moreover, the use of different strength criteria has a greater impact on the calculation results^[4]. Effect of strength theory in elastic-plastic analysis cannot be ignored, so finding a reasonable

strength criterion of geotechnical materials and structures is a research hotspot in geotechnical engineering.

2 Analysis of unified strength theory

Yu Maohong has carried out a systematic study of strength theory and proposed the unified strength theory on the basis of the double-shear stress yield criterion, the double-shear strength theory and the generalized double-shear stress yield criterion^[5]. The unified strength theory believes that the yield and failure of materials depend on the shear stress and normal stress on the section of the double-shear element, and when the two larger principal shear stresses and the corresponding normal stress on the double-shear element reach at the limit value, the material begins to yield or fail.

The mathematical expression is expressed as follows:

$$F = \tau_{13} + b\tau_{12} + A(\sigma_{13} + b\sigma_{12}) = C \quad (1)$$

for $\tau_{12} + A\sigma_{12} \geq \tau_{23} + A\sigma_{23}$

$$F' = \tau_{13} + b\tau_{23} + A(\sigma_{13} + b\sigma_{23}) = C \quad (2)$$

for $\tau_{12} + A\sigma_{12} \leq \tau_{23} + A\sigma_{23}$

Strength parameters of geotechnical materials are cohesion and internal friction angle, and the mathematical expressions of the unified strength theory expressed by principal stress and strength parameters are described as follows:

$$F = \sigma_1 - \frac{1 - \sin \varphi}{(1 + b)(1 + \sin \varphi)} (b\sigma_2 + \sigma_3) = \frac{2c \cos \varphi}{1 + \sin \varphi} \quad (3)$$

for $\sigma_2 \leq \frac{1}{2}(\sigma_1 + \sigma_3) + \frac{\sin \varphi}{2}(\sigma_1 - \sigma_3)$

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$$F' = \frac{1}{1+b}(\sigma_1 + b\sigma_2) - \frac{1 - \sin \varphi}{1 + \sin \varphi} \sigma_3 = \frac{2c \cos \varphi}{1 + \sin \varphi} \quad (4)$$

for $\sigma_2 \geq \frac{1}{2}(\sigma_1 + \sigma_3) + \frac{\sin \varphi}{2}(\sigma_1 - \sigma_3)$

In above formula c is cohesion, φ is internal friction angle and b is the strength criterion parameter.

The mathematical expressions of the unified strength theory expressed by stress invariant are described as follows:

$$F = \frac{I_1}{3}(1-\alpha) + (2+\alpha) \frac{\sqrt{J_2}}{\sqrt{3}} \cos \theta + \frac{\alpha(1-b)}{1+b} \sqrt{J_2} \sin \theta = \sigma_t \quad (5)$$

for $0^\circ \leq \theta \leq \theta_b$

$$F' = \frac{I_1}{3}(1-\alpha) + \left(\frac{2-b}{1+b} + \alpha\right) \frac{\sqrt{J_2}}{\sqrt{3}} \cos \theta + \left(\alpha + \frac{b}{1+b}\right) \sqrt{J_2} \sin \theta = \sigma_t \quad (6)$$

for $\theta_b \leq \theta \leq 60^\circ$

In which, $\alpha = \frac{1 - \sin \varphi}{1 + \sin \varphi}$, $\sigma_t = \frac{2c \cos \varphi}{1 + \sin \varphi}$, $\sigma_c = \frac{2c \cos \varphi}{1 - \sin \varphi}$,

θ_b is the stress angle of the boundary.

Intermediate principal stress has taken unified strength theory into consideration by introducing the parameter b , which has been verified that respectively the yield criterion as $b = 0$ is an inner envelope and the yield criterion as $b = 1$ is an outer one in π plane for stable material^[6].

Unified strength theory is a major breakthrough in the history of strength theory research. It is not a single strength criterion, but a collection of a series of strength criteria, covering the continuous space by changing the parameter b . Many strength criteria are special cases of unified strength theory. Therefore, unified strength theory has been widely used in geotechnical engineering (i.e., soil pressure, foundation bearing capacity, slope stability, underground engineering stability)^[7].

There are singularities on the yield surface of unified strength theory and which makes numerical calculations inconvenient. Mathematical processing of singularities is troublesome^[8]. In order to deal with the singularity of Mohr-Coulomb criterion, a series of Drucker-Prager criteria are proposed^[9, 10]. At present, Drucker-Prager criteria are applied to numerical calculation software (e.g., ABAQUS, ANSYS, FLAC, MARC).

3 Equivalent area circle yield criteria

3.1 Derivation of the general equation

Unified strength theory expressed in the Haigh-Westergaard stress space (ξ, ρ, θ) is detailed as follows:

$$F = \frac{\sqrt{3}}{3} \cdot \xi \cdot (1-\alpha) + (2+\alpha) \cdot \frac{\rho}{\sqrt{6}} \cdot \cos(\theta) + \frac{\alpha \cdot (1-b)}{1+b} \cdot \frac{\rho}{\sqrt{2}} \cdot \sin(\theta) - \sigma_t = 0 \quad (7)$$

for $0^\circ \leq \theta \leq \theta_b$

$$F' = \frac{\sqrt{3}}{3} \cdot \xi \cdot (1-\alpha) + \left(\frac{2-b}{1+b} + \alpha\right) \cdot \frac{\rho}{\sqrt{6}} \cdot \cos(\theta) + \left(\alpha + \frac{b}{1+b}\right) \cdot \frac{\rho}{\sqrt{2}} \cdot \sin(\theta) - \sigma_t = 0 \quad (8)$$

for $\theta_b \leq \theta \leq 60^\circ$

On the π plane, the vector radius with different stress angles can be easily obtained.

When $\theta = 0$, the conventional triaxial compression strength r_c is:

$$r_c = \frac{\sigma_t \cdot 6}{\sqrt{6} \cdot (\alpha + 2)} \quad (9)$$

When $\theta = \pi/3$, the conventional triaxial extension strength r_t is:

$$r_t = \frac{\sigma_t \cdot 6}{\sqrt{6} \cdot (2 \cdot \alpha + 1)} \quad (10)$$

When $\theta = \theta_b = \tan^{-1} \frac{\sqrt{3}}{2\alpha+1}$, the vector radius r_b is:

$$r_b = \frac{\sigma_t \cdot 3 \cdot (2 \cdot \alpha + 1) \cdot (b + 1) \cdot \sqrt{\frac{3}{(2 \cdot \alpha + 1)^2 + 1}}}{\sqrt{6} \cdot ((\alpha^2 + \alpha + 1) \cdot b + \alpha^2 + 4 \cdot \alpha + 1)} \quad (11)$$

The equation of equal area:

$$\frac{1}{6} \cdot \pi \cdot r_e^2 = \frac{1}{2} \cdot r_c \cdot r_b \cdot \sin(\theta_b) + \frac{1}{2} \cdot r_b \cdot r_t \cdot \sin\left(\frac{\pi}{3} - \theta_b\right) \quad (12)$$

By solving the above equation, we can get:

$$r_e = \sqrt{\frac{6}{\pi} \cdot \frac{3 \cdot \sigma_t^2 \cdot (b + 1) \cdot \sqrt{\frac{4 \cdot \alpha^2 + 4 \cdot \alpha + 4}{(2 \cdot \alpha + 1)^2 + 1}}}{2 \cdot (\alpha + 2) \cdot ((\alpha^2 + \alpha + 1) \cdot b + \alpha^2 + 4 \cdot \alpha + 1)} \cdot \sqrt{\left((\alpha + 2) \cdot \sin\left(\frac{\pi - 3 \cdot \theta_b}{3}\right) + (2 \cdot \alpha + 1) \cdot \sin(\theta_b)\right)}} \quad (13)$$

The equal area yield criterion can be written as:

$$F = \alpha \cdot I_1 + \sqrt{J_2} - k = 0 \quad (14)$$

In which, $k = \frac{r_e}{\sqrt{2}}$, $\alpha = k \cdot \frac{\sin(\varphi)}{3 \cdot c \cdot \cos(\varphi)}$.

It is interesting that the form of the equal area yield criterion is the same as Drucker-Prager criteria, while the parameters are different.

Five Drucker-Prager criteria are proposed base on Mohr-Coulomb criterion (M-C) in geotechnical engineering which include M-C exterior angle circumcircle, M-C interior angle circumcircle, M-C inscribed circle, M-C equivalent area circle, M-C non-associated matching circle.

The analytical expressions of parameters α and k of yield criteria (DP1~5 and the new criteria) are given in Table 1.

Table1. Parameters' expressions of yield criteria

yield criteria	α	k
DP1	$\frac{2 \cdot \sin(\varphi)}{\sqrt{3} \cdot (3 - \sin(\varphi))}$	$\frac{(6 \cdot c \cdot \cos(\varphi))}{\sqrt{3} \cdot (3 - \sin(\varphi))}$
DP2	$\frac{2 \cdot \sin(\varphi)}{\sqrt{3} \cdot (3 + \sin(\varphi))}$	$\frac{6 \cdot c \cdot \cos(\varphi)}{\sqrt{3} \cdot (3 + \sin(\varphi))}$
DP3	$\frac{\sin(\varphi)}{\sqrt{3} \cdot \sqrt{\sin(\varphi)^2 + 3}}$	$\frac{\sqrt{3} \cdot c \cdot \cos(\varphi)}{\sqrt{(4 - \cos(\varphi)^2)}}$
DP4	$\frac{2 \cdot \sqrt[4]{3} \cdot \sin(\varphi)}{\sqrt{2 \cdot \pi \cdot (9 - (\sin(\varphi))^2)}}$	$\frac{6 \cdot \sqrt[4]{3} \cdot c \cdot \cos(\varphi)}{\sqrt{2 \cdot \pi \cdot (9 - (\sin(\varphi))^2)}}$
DP5	$\frac{\sin(\varphi)}{3}$	$c \cdot \cos(\varphi)$
New	$\frac{2 \cdot \sin(\varphi)}{1 + \sin(\varphi)} \cdot T$	$\frac{6 \cdot c \cdot \cos(\varphi)}{1 + \sin(\varphi)} \cdot T$

Note:

- DP1: M-C exterior angle circumcircle
- DP2: M-C interior angle circumcircle
- DP3: M-C inscribed circle
- DP4: M-C equivalent area circle
- DP5: M-C non-associated matching circle
- New: Equivalent area circle yield criteria

$$T = \frac{1}{\sqrt{\pi}} \cdot \frac{(b+1) \cdot \left((\alpha+2) \cdot \sin\left(\frac{\pi}{3} - \theta_b\right) + (2 \cdot \alpha + 1) \cdot \sin(\theta_b) \right) \cdot \sqrt{\alpha^2 + \alpha + 1}}{(\alpha+2) \cdot \left((\alpha^2 + \alpha + 1) \cdot b + \alpha^2 + 4 \cdot \alpha + 1 \right) \cdot (2 \cdot \alpha + 1)}$$

3.2 Calculation and analysis

The general equation is not a single criterion, and it contains a series of new criteria.

For $c=0$, the value of parameter α calculated by analytical expressions are given in Table 2.

Table2. Comparison of parameter α calculated by different yield criteria ($c = 0$)

yield criteria	internal friction angle (degree)					
	10	20	30	40	50	60
DP1	0.0709	0.149	0.231	0.315	0.396	0.469
DP2	0.0632	0.118	0.165	0.204	0.235	0.259
DP3	0.0576	0.112	0.160	0.201	0.234	0.258
DP4	0.0609	0.121	0.177	0.230	0.277	0.317
DP5	0.0579	0.114	0.167	0.214	0.255	0.289
New $b=0$	0.0609	0.121	0.177	0.230	0.277	0.317
New $b=0.1$	0.0623	0.123	0.181	0.234	0.281	0.319

New $b=0.2$	0.0635	0.125	0.184	0.237	0.283	0.321
New $b=0.25$	0.0641	0.126	0.185	0.239	0.285	0.322
New $b=0.3$	0.0646	0.127	0.187	0.240	0.286	0.323
New $b=0.4$	0.0657	0.129	0.189	0.242	0.288	0.324
New $b=0.5$	0.0666	0.131	0.191	0.245	0.290	0.326
New $b=0.6$	0.0674	0.132	0.193	0.247	0.292	0.327
New $b=0.7$	0.0681	0.134	0.195	0.248	0.293	0.328
New $b=0.75$	0.0685	0.134	0.195	0.249	0.294	0.329
New $b=0.8$	0.0688	0.135	0.196	0.250	0.295	0.329
New $b=0.9$	0.0695	0.136	0.198	0.252	0.296	0.330
New $b=1$	0.0701	0.137	0.199	0.253	0.297	0.331

As seen from table 2, the value of parameter α of DP4 is the same as of the new criterion for $b = 0$. It shows that the new criterion is reliable because the unified strength theory degenerates to Mohr-Coulomb criterion when $b = 0$. The surface of unified strength theory is convex when $0 < b < 1$, and it is non-convex when $b > 1$ or $b < 0$. While the new series of equivalent area circle yield criteria based on unified strength theory are always convex.

Some equations can be derived base on UST (unified strength theory) exterior angle circumcircle, UST interior angle circumcircle, UST middle angle circumcircle, UST inscribed circle, etc.

The new parameters α_e can be written as: $\alpha_e = t \cdot \alpha$, in which t is the area parameter. Therefore, the new general equation of a series of equivalent area circle yield criteria based on unified strength theory essentially includes these above equations by changing the value of the parameter t .

But it is unnecessary for which can lead to overestimate or underestimate of the strength of the material.

The flow vector of the strength criterion is:

$$\frac{\partial F}{\partial \sigma_{ij}} = C_1 \cdot \frac{\partial I_1}{\partial \sigma_{ij}} + C_2 \cdot \frac{\partial \sqrt{J_2}}{\partial \sigma_{ij}} + C_3 \cdot \frac{\partial J_3}{\partial \sigma_{ij}} \quad (15)$$

In which: $C_1 = \alpha_e$, $C_2 = 1$, $C_3 = 0$,

$$\frac{\partial I_1}{\partial \sigma_{ij}} = [1, 1, 1, 0, 0, 0]^T,$$

$$\frac{\partial \sqrt{J_2}}{\partial \sigma_{ij}} = \frac{1}{2\sqrt{J_2}} [S_x, S_y, S_z, 2S_{xy}, 2S_{yz}, 2S_{zx}]^T,$$

$$\frac{\partial J_3}{\partial \sigma_{ij}} = \begin{bmatrix} (S_y S_z - S_{yz}^2) + \frac{1}{3} J_2 \\ (S_x S_z - S_{xz}^2) + \frac{1}{3} J_2 \\ (S_x S_y - \tau_{xy}^2) + \frac{1}{3} J_2 \\ 2(S_{yz} S_{xz} - S_z S_{xy}) \\ 2(S_{xz} S_{xy} - S_x S_{yz}) \\ 2(S_{xy} S_{yz} - S_y S_{xy}) \end{bmatrix}.$$

The form of the new equation is similar to Drucker-Prager criterion. It is a cluster of strength criteria, rather than a single strength criterion, and Drucker-Prager criterion is a special case of the new strength criteria. The new yield criteria have no singularities; thus, it is suitable for numerical calculations. By changing the parameter b , a series of continuously changing strength criteria can be obtained.

4 Conclusion

Based on the in-depth analysis of existing strength criteria, a series of new strength criteria are proposed. And the parameters of new strength criteria are calculated and discussed in detail. The coefficients of the flow vector are given, which can be used to develop subroutines of finite element software for numerical calculation.

The new strength criteria are a series of equivalent area circle yield criteria based on unified strength theory through the continuous change of b . The new strength criteria have no singularities and which are smooth everywhere on the yield surface.

The equivalent area circle yield criteria are very convenient for numerical calculations, which can provide an important reference for evaluating the effect of strength theory.

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