https://doi.org/10.1051/e3sconf/202130403019

Refining the Galerkin method error estimation for parabolic type problem with a boundary condition

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Abstract. The article considers a parabolic-type boundary value problem with a divergent principal part, when the boundary condition contains the time derivative of the required function:

$$\begin{cases} u_t - \dfrac{d}{dx_i} a_i(x,t,u,\nabla u) + a(x,t,u,\nabla u) = 0\,, \\ a_0 u_t + a_i(x,t,u,\nabla u)\cos(\nu, x_i) = g(x,t,u), \quad (x,t) \in S_t\,, \\ u(x,0) = u_0(x)\,, \quad x \in \Omega \end{cases}$$

Such nonclassical problems with boundary conditions containing the time derivative of the desired function arise in the study of a number of applied problems, for example, when the surface of a body, whose temperature is the same at all its points, is washed off by a well-mixed liquid, or when a homogeneous isotropic body is placed in the inductor of an induction furnace and an electro-magnetic wave falls on its surface. Such problems have been little studied, therefore, the study of problems of parabolic type, when the boundary condition contains the time derivative of the desired function, is relevant. In this paper, the definition of a generalized solution of the considered problem in the space $\widetilde{\mathrm{H}^{1,1}}(Q_T)$is given. This problem is solved by the approximate Bubnov-Galerkin method. The coordinate system is chosen from the space $H^1(\Omega)$. To determine the coefficients of the approximate solution, the parabolic problem is reduced to a system of ordinary differential equations. The aim of the study is to obtain conditions under which the estimate of the error of the approximate solution in the norm $H^1(\Omega)$ has order $O(h^{k-1})$. The paper first explores the auxiliary elliptic problem. When the condition of the ellipticity of the problem is satisfied, inequalities are proposed for the difference of the generalized solution of the considered parabolic problem with a divergent principal part, when the boundary condition contains the time derivative of the desired function and the solution of the auxiliary elliptic problem. Using these estimates, as well as under additional conditions for the coefficients and the function included in the problem under consideration, estimates of the error of the approximate solution of the Bubnov-Galerkin method in the norm $H^1(\Omega)$ of order $O(h^{k-1})$ for the considered nonclassical parabolic problem with divergent principal part, when the boundary condition contains the time derivative of the desired function.

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1 Introduction

When studying a number of topical technical problems, it becomes necessary to study mixed problems of parabolic type, when the boundary condition contains the time derivative of the desired function. Problems of this type occur, for example, when a homogeneous isotropic body is placed in the inductor of an induction furnace and an electromagnetic wave is incident on its surface. Some nonlinear problems parabolic with the boundary condition containing the time derivative of the desired function has been viewed, for example, in papers [1-3]. Many scientists were engaged in the construction of an approximate solution by the Galerkin method and obtaining a priori estimates for an approximate solution for parabolic classical quasilinear problems without a time derivative in the boundary condition: Mikhlin S.G., Douglas J. Jr, Dupont T., Dench JE, Jr, Jutchell L and other [4-9]. And quasilinear problems, when the boundary condition contains the time derivative of the required function using the Galerkin method, were studied in [10-13].

2 Methods

In this paper, we consider a quasilinear problem of parabolic type, when the boundary condition contains the time derivative of the required function:

$$\begin{cases} u_t - \frac{d}{dx_i} a_i(x,t,u,\nabla u) + a(x,t,u,\nabla u) = 0 \quad , \\ a_0 u_t + a_i(x,t,u,\nabla u)\cos(\nu, x_i) = g(x,t,u), \quad (x,t) \in S_t \ , \\ u(x,0) = u_0(x) \ , \ x \in \Omega \end{cases} \tag{1}$$

where $\Omega - bounded\ domain\ in\ \mathrm{E}_m, m =$ dim $-$ the dimension of the domain Ω,

$$Q_T = \{\Omega \times [0,T]\}, \qquad S_T = \{\partial\Omega \times [0,T]\}, \qquad a_0 = const > 0$$

A generalized solution from the space $\widetilde{\mathrm{H}^{1,1}}(Q_T) = \{u \in H^{1,1}(Q_T) : a_0 u_t \in L_2(S_T)\}$ to problem (1) is a function from $\widetilde{\mathrm{H}^{1,1}}(Q_T)$ satisfying the identity

$$\int_{Q_T} \left(u_t \eta + a_i(x,t,u,\nabla u)\eta_{x_i} + a(x,t,u,\nabla u)\eta\right) dxdt + \\ + \int_{S_T} (a_0 u_t + g(x,t,u))\eta \, dxdt = 0 \tag{2}$$

$$\forall\, \eta \in H^{1,1}(Q_T)$$

We construct an approximate solution of Galerkin [14-16]. We take a coordinate system from the space $H^1(\Omega)$. We will seek an approximate solution U (x, t) in the form
$U(x,t) = \sum_{k=1}^{n} C_k^n(t)\varphi_k(x)$

where $\mathrm{C}_k^n(t)$ are determined from the system of ordinary differential equations

$$(U_t, \varphi_j)_{\hat{L}_2} + (a_i(x,t,U,\nabla U), \varphi_{jx_i})_\Omega + (a(x,t,U,\nabla U), \varphi_j)_\Omega = \\ = (g(x,t,U), \varphi_j)_S \ , \quad j = \overline{1,n} \tag{3}$$

with initial conditions

$$(U(x,O) - u_0, \varphi_j)_{H^1(\Omega)} = 0$$

Here $\hat{L}_2(\Omega)$ – is the space of a function with scalar product
$(u,v)_{\hat{L}_2} = (u,v)_\Omega + a_0(u,v)_s$

The purpose of this article is to find out the conditions under which the estimate of the error of the approximate solution in the norm $H^1(\Omega)$ has order $O(h^{k-1}), 1 \leq k \leq r$ we assume that the set $M = M_h$ is chosen from the family S_h^r.

3 Results

We first investigate the following elliptic problem: Find a function $W(x,t) \in M$ satisfying the integral identity [17-18]

$$\left(a_i(x,t,u,\nabla u) - a_i(x,t,u,\nabla W), v_{x_i}\right)_\Omega + \lambda(u - W, v)_\Omega = (g(x,t,u) - -g(x,t,W), v)_s, \quad \forall v \epsilon M \tag{4}$$

where u is the solution to problem (2).

Suppose that λ is such a sufficiently large positive number that problem (3) has a unique solution.

In what follows, assume that the following norms are bounded [19-21] $\|u\|_{\left(L_\infty\left(o,T;L_\infty(\overline{\Omega})\right)\right)}$, $\|\nabla u\|_{(L_\infty(o,T;L_\infty(\Omega)))}$, $\|W\|_{\left(L_\infty\left(o,T;L_\infty(\overline{\Omega})\right)\right)}$, ≤ $\|\nabla W\|_{\left(L_\infty\left(o,T;L_\infty(\overline{\Omega})\right)\right)} \leq К = const,$

Let $u \in H^k(\Omega), 1 \leq k \leq r$, and $W(x,t)$be solutions to problem (2) and (3), respectively. The set $M = M_h$ selected from the family S_h^r. In addition, let the condition be satisfied:

$$(p_i - q_i)(a_i(x,t,u,\mathrm{p}) - a_i(x,t,u,\mathrm{q})) \geq c(p-q)^2 \text{ , c=const} > 0 \tag{5}$$

and the functions $a_i(x,t,u,\nabla u)$ satisfy the Lipschitz condition in ∇u, and the functions $g(x,t,u)$- in u. Then the inequality is true

$$\|\eta\|_{H'(\Omega)} \leq C\|u\|_{H^k(\Omega)} h^{k-1}, \qquad 1 \leq k \leq r \tag{6}$$

where $\eta = u - W$

In addition, let the solution of problem (2) satisfy

$$u(x,t) \in L_{\bar{q}}\left(O,T,H^k(\Omega)\right), \tag{7}$$

$$u_t \in L_2\left(O,T,H^k(\Omega)\right) \cap L_{\bar{p}}\left(O,T,L_{\tilde{p}}(\Omega)\right) \cap L_{\bar{p}}\left(O,T,W_q^1(\Omega)\right), \qquad 1 \leq k \leq r \tag{8}$$

where

$$\bar{p} = \frac{2\bar{q}}{\bar{q} - 2\gamma}, \qquad \bar{q} \geq 2; \qquad q = 2 + \frac{2\gamma}{1-\gamma};$$

$$0 < \gamma \leq \alpha \leq 1, \qquad \tilde{p} = \frac{2\tilde{q}}{\tilde{q} - 2\gamma},$$

$$\tilde{q} \in \begin{cases} \left[2, \frac{2(m-1)}{m-2}\right], m \geq 3 \\ [2,\infty) \qquad m = 2 \end{cases} \tag{9}$$

And let conditions (4) and the ellipticity condition be satisfied

$$v\sum_{i=1}^{m} p_i^2 \leq \frac{\partial a_i}{\partial p_j}\ p_i p_j \leq \mu \sum_{i=1}^{n} p_i^2, \quad \forall p \in E_m \tag{10}$$

In addition, we assume that the functions $\frac{\partial a_i}{\partial t}, \frac{\partial a_i}{\partial u}, \frac{\partial a_i}{\partial p}$ satisfy the Hölder condition with respect to ∇u with exponent α, and the functions $\frac{\partial g}{\partial t}, \frac{\partial g}{\partial u}$ - in u.

If the set $M = M_h$ belongs to the family S_h^r, then the estimate occur

$$\left\|\frac{\partial \eta}{\partial t}\right\|_{L_2\left(O,T,H^1(\Omega)\right)} \leq Ch^{\gamma(k-1)} \tag{11}$$

where $C = C(R)\left(1+\left\|\frac{\partial u}{\partial t}\right\|_{L_{\bar{p}}\left(O,T,L_{\widetilde{p}}(S)\right)}+\left\|\frac{\partial u}{\partial t}\right\|_{L_{\bar{p}}\left(O,T,W_q^1(\Omega)\right)}\right) \times \left(\|u\|_{L_{\bar{q}}\left(O,T,H^k(\Omega)\right)}+\left\|\frac{\partial u}{\partial t}\right\|_{L_2\left(O,T,H^k(\Omega)\right)}\right)$ (12)

Theorem. Let $u, \cup, W$ be solutions to problems (2), (3) and (4), respectively. Suppose that conditions (5) - (10) are satisfied. Suppose also that on the set

$$\left\{(x,t) \in \bar{Q}_T, \|v\|_{L_\infty(0,T,L_\infty(\bar{\Omega}))}, \|\nabla v\|_{L_\infty(0,T,L_\infty(\Omega))} \le R\right\}$$

the functions $\frac{\partial a_i}{\partial t}(x,t,v,\nabla v)$ are bounded, the function $a(x,t,v,\nabla v)$ satisfies the Lipschitz condition with respect to v with respect to ∇v, and $g(x,t,v)$ - with respect to v, moreover, let $U(x,o)$ satisfy the inequality

$$\|u(x,0) - \cup(x,0)\|_{\widetilde{L_2}} \le Ch^{(k-1)},$$

Then, the following estimate is true

$$\|u - \cup\|_{L_\infty(O,T,\widetilde{L_2})} + \|u - \cup\|_{L_2(O,T,H^1(\Omega))} \le Ch^{\gamma(k-1)}, 1 \le k \le r \tag{13}$$

relation (12) is true for C.

Proof. Let us integrate equation (3) from 0 to t. Subtract the resulting equation from equation (2). Suppose that $\zeta = \eta + \xi$ and taking into account equation (4), we arrive at the identity

$$\int_0^t (\xi_t, v)_{\widetilde{L_2}} dt + \int_0^t \left(a_i(x,t,U,\nabla W)dt + a_i(x,t,U,\nabla U), v_{x_i}\right)dt$$
$$= \int_0^t \left(a_i(x,t,U,\nabla W) + a_i(x,t,u,\nabla W), v_{x_i}\right)_\Omega dt \tag{14}$$
$$- \int_0^t \{\left(\frac{\partial \eta}{\partial t}, v\right)_{\widetilde{L_2}} + (a(x,t,u,\nabla u) - a(x,t,U,\nabla U), v)_\Omega + \lambda(\eta, v)_\Omega$$
$$- (g(x,t,W) - g(x,t,U), v) \quad _S\}\, dt, \qquad \forall v \in M$$

For a test function v we take $v = \xi(\cdot,t)$ The left-hand side of equality 14) is estimated from below using assumption (5)

$$\int_0^t \left(a_i(x,t,U,\nabla W) - a_i(x,t,U,\nabla U), \xi_{x_i}\right)_\Omega dt \ \ge\ \gamma_1 \|\nabla\xi\|^2_{L_2(O,t,L_2(\Omega))}$$

We estimate the terms on the right-hand side of (14) from above using the Cauchy inequalities and the assumptions of the theorem as follows

$$\int_0^t \left|\left(a_i(x,t,\cup,\nabla W) - a_i(x,t,u,\nabla W), \xi_{x_i}\right)_\Omega\right| dt \le$$

$$\le \boldsymbol{\varepsilon}\|\nabla\xi\|^2_{L_2(0,t,L_2(\Omega))} + C\int_0^t \int_\Omega \ [a_i(x,t,\cup,\nabla W) - a_i(x,t,u,\nabla W)]^2 dxdt \le$$

$$\le \boldsymbol{\varepsilon}\|\nabla\xi\|^2_{L_2(0,t,L_2(\Omega))} + C\left(\|\eta\|^2_{L_2(0,t,L_2(\Omega))} + \|\xi\|^2_{L_2(0,t,L_2(\Omega))}\right),$$

$$\int_0^t |(a(x,t,u,\nabla u) - a(x,t,\cup,\nabla\cup), \xi)_\Omega|\, dt \le C\left(\|\xi\|^2_{L_2(0,t,L_2(\Omega))} + \|\eta\|^2_{L_2\left(0,t,H^1(\Omega)\right)}\right) +$$
$$+\boldsymbol{\varepsilon}\|\nabla\xi\|^2_{L_2(0,t,L_2(\Omega))}$$

Similarly, we have

$$\left|\int_0^t \int_s \left(g(x,t,W) - g(x,t,\cup)\right)\xi dxdt\right| \leq C\int_0^t \int_s \xi^2 dxdt \leq \boldsymbol{\varepsilon}\|\nabla\xi\|^2_{L_2(0,t,L_2(\Omega))} +$$

$$+C\|\xi\|^2_{L_2(0,t,L_2(\Omega))}$$

The rest of the terms are estimated in the same way. Next, we take $\varepsilon = \frac{\gamma_1}{6}$. We substitute the obtained estimates in (14). Then, after reducing similar terms and taking into account the conditions of the theorem and inequalities (6) and (11), we derive the desired estimate (13). The theorem is proved.

4 Conclusions

When the condition of the ellipticity of the problem is satisfied, inequalities are proposed for the difference of the generalized solution of the considered parabolic problem with a divergent principal part and the solution of the auxiliary elliptic problem. Using these estimates and under additional conditions on the coefficients of problem (1), we obtain estimates of the error of the approximate solution of the Bubnov-Galerkin method in the norm $H^1(\Omega)$ of order $O(h^{k-1})$ for the considered nonclassical parabolic problem with divergence the main part, when the boundary condition contains the time derivative of the desired function.

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