Finite element modeling of integral viscoelastic properties of textile composites

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Abstract. The problem of modeling the effective integral viscoelastic properties of unidirectional composite materials is considered. To calculate the integral properties of viscoelasticity, the Fourier transform and the inverse Fourier transform are used, as well as the method of asymptotic averaging for composites with steady polyharmonic vibrations, and a finite element algorithm for solving local problems of the viscoelasticity theory on the periodicity cell of the composite. To obtain the material constants, a method of approximation of the Fourier images of the relaxation and creep kernels is proposed, which makes it possible to avoid the numerical error of the inverse Fourier transform. **Key words:** unidirectional composite materials, viscoelastic properties, numerical modeling.

1 Introduction

At present, polymer composite materials (PCM) are widely used in various industries, which exhibit significant viscoelastic properties [1–8], especially during long-term operation. PCM are used in the development of damping structures for the aviation, shipbuilding and automotive industries. In the products of these industries, PCM, as a rule, are operated for a long time - several years and even decades, so the problem of predicting creep deformations of PCM structures for these industries is relevant.

Various methods for calculating the viscoelastic properties of composites, mainly for cyclic vibrations, were proposed in [1, 2, 9–13]. For such structures, the calculation of creep deformations of composites, which manifests itself over a long period of time, is of great importance. Also important is the dependence of this process on the content of their structural components - reinforcing fibers, dispersed particles, etc., including in situations where the fibers of one composite material can themselves be considered as a composite material. To solve this problem, it is necessary to calculate the effective relaxation and creep kernels of a composite that may contain anisotropic components.

Methods for solving the problem of calculating creep deformations were considered in [9, 10, 12, 14, 15]. These methods are mainly based on approximate algorithms for calculating effective viscoelasticity operators and applying the Laplace transform to invert these operators. In [16], to calculate the viscoelasticity operators of composites, it was proposed to use the method of asymptotic averaging, and to invert these operators, a

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generalization of the approximation method by A.A. Ilyushin [17]. Various versions of the method for applying the averaging method for calculating the stress-strain state of thin viscoelastic structures were proposed in [18–22].

The purpose of this article is to develop an algorithm for calculating the effective relaxation and creep kernels of unidirectional composites and their subsequent approximation in the form of exponential kernels. It is shown that this method provides high accuracy of calculations of the relaxation and creep kernels of unidirectional composites.

2 Statement of the linear viscoelasticity problem under harmonic loading

Consider the problem of mechanics of a linear viscoelastic continuum in the integral formulation [23]:

$$\begin{cases} \nabla \cdot \boldsymbol{\sigma} = \mathbf{0}, \\ \boldsymbol{\varepsilon} = \dfrac{1}{2}\left(\nabla \otimes \mathbf{u} + (\nabla \otimes \mathbf{u})^T\right), \\ \boldsymbol{\sigma} = {}^4\mathbf{C} \cdot\cdot\, \boldsymbol{\varepsilon} - \displaystyle\int_0^t {}^4\mathbf{K}(t-\tau) \cdot\cdot\, \boldsymbol{\varepsilon}(\tau) d\tau, \\ \Sigma_\sigma : \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{S}^e(t), \\ \Sigma_u : \mathbf{u} = \mathbf{u}^e(t). \end{cases} \tag{1}$$

Here are denoted: $\boldsymbol{\sigma}$ – stress tensor, $\boldsymbol{\varepsilon}$ – Cauchy small strain tensor, $\mathbf{u}$ – displacement vector, ${}^4\mathbf{C}$ – fourth-order stiffness tensor (elastic moduli tensor), ${}^4\mathbf{K}(t)$ – relaxation kernel tensor, $\mathbf{S}^e(t)$ – vector of external forces on part Σ_σ of the boundary, $\mathbf{u}^e(t)$ – vector of specified displacements on a part Σ_u of the boundary, $\mathbf{n}$ – outward normal vector.

The first equation in the system (1) is the equilibrium equation, the second is the Cauchy relations, the third is the constitutive relations of the linear theory of viscoelasticity in the Volterra form [17, 23], the fourth and fifth are the boundary conditions.

Consider the case of steady oscillations, when the change in the vectors of external forces $\mathbf{S}^e(t)$ and displacements $\mathbf{u}^e(t)$ on the boundary obey the monoharmonic law

$$\mathbf{S}^e(t) = \mathrm{Re}\left(\mathbf{S}^{e*} e^{i\omega t}\right), \qquad \mathbf{u}^e(t) = \mathrm{Re}\left(\mathbf{u}^{e*} e^{i\omega t}\right), \tag{2}$$

where $\mathbf{S}^{e*}$, $\mathbf{u}^{e*}$ are the complex oscillation amplitudes, ω is the specified oscillation frequency. Then the displacements $\mathbf{u}$, which are the solution to problem (1), are considered to have the monoharmonic form

$$\mathbf{u}(\mathbf{x},t) = \mathrm{Re}\left(\mathbf{u}^*(\mathbf{x})e^{i\omega t}\right). \tag{3}$$

Substituting (3) into system (1), and also taking into account (2), it can be shown that in this case, strains and stresses also change according to the monoharmonic law

$$\boldsymbol{\varepsilon} = \mathrm{Re}\left(\boldsymbol{\varepsilon}^* e^{i\omega t}\right), \quad \boldsymbol{\sigma} = \mathrm{Re}\left(\boldsymbol{\sigma}^* e^{i\omega t}\right), \tag{4}$$

and the constitutive relations take the form [17, 23]

$$\boldsymbol{\sigma}^* = {}^4\mathbf{C}^*(\omega)\cdot\cdot\,\boldsymbol{\varepsilon}^*. \tag{5}$$

where ${}^4\mathbf{C}^*(\omega)$ is the tensor of complex elastic moduli

$${}^4\mathbf{C}^*(\omega) = {}^4\mathbf{C} - {}^4\mathbf{K}^*(\omega). \tag{6}$$

Here ${}^4\mathbf{K}^*(\omega)$ is the tensor of complex relaxation kernels and it is defined as follows:

$${}^4\mathbf{K}^*(\omega) = \int_0^\infty {}^4\mathbf{K}(\tau)e^{-i\omega\tau}d\tau. \tag{7}$$

Consider, in addition to the direct defining relation in system (1), also the inverse relation to it

$$\boldsymbol{\varepsilon} = {}^4\boldsymbol{\Pi}\cdot\cdot\,\boldsymbol{\sigma} + \int_0^t {}^4\boldsymbol{\Gamma}(t-\tau)\cdot\cdot\,\boldsymbol{\sigma}(\tau)d\tau, \tag{8}$$

where ${}^4\boldsymbol{\Pi}$ is the elastic compliance tensor and ${}^4\boldsymbol{\Gamma}(t)$ is the creep kernel tensor. Then, for harmonic vibrations, we obtain the constitutive relations in the following form:

$$\boldsymbol{\varepsilon}^* = {}^4\boldsymbol{\Pi}^*(\omega)\cdot\cdot\,\boldsymbol{\sigma}^*, \tag{9}$$

where ${}^4\boldsymbol{\Pi}^*(\omega)$ is the complex elastic compliance tensor

$${}^4\boldsymbol{\Pi}^*(\omega) = {}^4\boldsymbol{\Pi} + {}^4\boldsymbol{\Gamma}^*(\omega). \tag{10}$$

Here ${}^4\boldsymbol{\Gamma}^*(\omega)$ is the tensor of complex creep kernels that is defined as follows:

$$ {}^4\mathbf{\Gamma}^*(\omega) = \int_0^\infty {}^4\mathbf{\Gamma}(\tau) e^{-i\omega\tau} d\tau. \tag{11} $$

Adding to these relations for the amplitudes the equations of quasi-static oscillations, as well as the boundary conditions, we obtain the following statement of the problem of the viscoelasticity theory for steady oscillations:

$$ \begin{cases} \nabla \cdot \boldsymbol{\sigma}^* = \mathbf{0}, \\ \boldsymbol{\varepsilon}^* = \dfrac{1}{2}\left(\nabla \otimes \mathbf{u}^* + (\nabla \otimes \mathbf{u}^*)^T \right), \\ \boldsymbol{\sigma}^* = {}^4\mathbf{C}^* \cdot\cdot \boldsymbol{\varepsilon}^*, \\ \Sigma_\sigma : \boldsymbol{\sigma}^* \cdot \mathbf{n} = \mathbf{S}^{e*}, \\ \Sigma_u : \mathbf{u}^* = \mathbf{u}^{e*}. \end{cases} \tag{12} $$

The system (12) will be used to find the effective integral viscoelastic characteristics of the 1D composite.

3 General form for creep and relaxation spectral functions and kernels

Let us represent the relaxation and creep functions using spectral expansions of rank 4 tensors for different symmetry classes of materials [23]

$$ \begin{aligned} {}^4\mathbf{R}(t) &= \sum_{\alpha,\beta=1}^{m} R_{\alpha\beta}(t) \frac{\mathbf{a}_{(\alpha)} \otimes \mathbf{a}_{(\beta)}}{a_{(\alpha)} a_{(\beta)}} + \sum_{\alpha=m+1}^{n} R_{\alpha\alpha}(t) {}^4\mathbf{\Gamma}_{(\alpha)}, \\ {}^4\mathbf{\Pi}(t) &= \sum_{\alpha,\beta=1}^{m} \Pi_{\alpha\beta}(t) \frac{\mathbf{a}_{(\alpha)} \otimes \mathbf{a}_{(\beta)}}{a_{(\alpha)} a_{(\beta)}} + \sum_{\alpha=m+1}^{n} \Pi_{\alpha\alpha}(t) {}^4\mathbf{\Gamma}_{(\alpha)}. \end{aligned} \tag{13} $$

Here m, n, $\mathbf{a}_{(\alpha)}$ ($\alpha = 1,...,m$), ${}^4\mathbf{\Gamma}_{(\alpha)}$ ($\alpha = m+1,...,n$) depend on the material symmetry class, $a_{(\alpha)} = (\mathbf{a}_{(\alpha)} \cdot\cdot \mathbf{a}_{(\alpha)})^{1/2}$ ($\alpha = 1,...,m$). Functions $R_{\alpha\beta}(t)$ and $\Pi_{\alpha\beta}(t)$ are called spectral relaxation and creep functions respectively. The spectral kernels of relaxation $K_{\alpha\beta}(t)$ and creep $\Gamma_{\alpha\beta}(t)$ are obtained from the relations:

$$
\begin{aligned}
{}^4\mathbf{K}(t) &= -\frac{\partial\, {}^4\mathbf{R}(t)}{\partial t}, \\
{}^4\mathbf{\Gamma}(t) &= \frac{\partial\, {}^4\mathbf{\Pi}(t)}{\partial t}
\end{aligned}
\tag{14}
$$

4 Spectral relaxation and creep functions and kernels for an isotropic material

Consider the spectral representation (13) for an isotropic symmetry class. In this case $m=1$, $n=2$, $\mathbf{a}_{(1)}=\mathbf{E}$, $a_{(1)}=\sqrt{3}$, ${}^4\mathbf{\Gamma}_{(2)}=\mathbf{\Delta}-\frac{1}{3}\mathbf{E}\otimes\mathbf{E}$. Here $\mathbf{\Delta}$ is the unit tensor of the 4th rank, $\mathbf{E}$ is the metric tensor [24]. Thus, for an isotropic symmetry class, the spectral representation of the relaxation and creep function tensors has the form

$$
\begin{aligned}
{}^4\mathbf{R}(t) &= R_K(t)\mathbf{E}\otimes\mathbf{E} + 2R_G(t)\left(\mathbf{\Delta}-\frac{1}{3}\mathbf{E}\otimes\mathbf{E}\right), \\
{}^4\mathbf{\Pi}(t) &= \Pi_K(t)\mathbf{E}\otimes\mathbf{E} + 2\Pi_G(t)\left(\mathbf{\Delta}-\frac{1}{3}\mathbf{E}\otimes\mathbf{E}\right),
\end{aligned}
\tag{15}
$$

Here are denoted: $R_K(t)=\frac{1}{3}R_{11}(t)$ – volumetric relaxation function; $\Pi_K(t)=\frac{1}{3}\Pi_{11}(t)$ – volumetric creep function; $R_G(t)=2R_{22}(t)$ – shear relaxation function; $\Pi_G(t)=2\Pi_{22}(t)$ – shear creep function. From formulas (14) and (15) we obtain

$$
\begin{aligned}
{}^4\mathbf{K}(t) &= K_K(t)\mathbf{E}\otimes\mathbf{E} + 2K_G(t)\left(\mathbf{\Delta}-\frac{1}{3}\mathbf{E}\otimes\mathbf{E}\right), \\
{}^4\mathbf{\Gamma}(t) &= \Gamma_K(t)\mathbf{E}\otimes\mathbf{E} + 2\Gamma_G(t)\left(\mathbf{\Delta}-\frac{1}{3}\mathbf{E}\otimes\mathbf{E}\right).
\end{aligned}
\tag{16}
$$

For most isotropic materials, the assumption of the absence of bulk relaxation and creep is true [16, 17, 23], i.e., $R_K(t)=0$, $\Pi_K(t)=0$, $K_K(t)=0$, $\Gamma_K(t)=0$, then the relaxation and creep kernels (16) are simplified:

$$
\begin{aligned}
{}^4\mathbf{K}(t) &= 2K_G(t)\left(\Delta - \frac{1}{3}\mathbf{E}\otimes\mathbf{E}\right),\\
{}^4\mathbf{\Gamma}(t) &= 2\Gamma_G(t)\left(\Delta - \frac{1}{3}\mathbf{E}\otimes\mathbf{E}\right).
\end{aligned}
\tag{17}
$$

Consider the complex moduli of elasticity and elastic compliance for an isotropic material. According to formulas (6) and (11), the tensors of complex relaxation and creep kernels take the form

$$
\begin{aligned}
{}^4\mathbf{K}^*(\omega) &= 2K_G^*(\omega)\left(\Delta - \frac{1}{3}\mathbf{E}\otimes\mathbf{E}\right),\\
{}^4\mathbf{\Gamma}^*(\omega) &= 2\Gamma_G^*(\omega)\left(\Delta - \frac{1}{3}\mathbf{E}\otimes\mathbf{E}\right),
\end{aligned}
\tag{18}
$$

where $K_G^*(\omega)$ and $\Gamma_G^*(\omega)$ are complex functions of shear relaxation and creep, respectively, determined by the formulas

$$
\begin{aligned}
K_G^*(\omega) &= \int_0^\infty K_G(\tau)e^{-i\omega\tau}d\tau,\\
\Gamma_G^*(\omega) &= \int_0^\infty \Gamma_G(\tau)e^{-i\omega\tau}d\tau.
\end{aligned}
\tag{19}
$$

From (6) and (18) an expression for the complex modulus of elasticity is obtained:

$$
{}^4\mathbf{C}^*(\omega) = K\mathbf{E}\otimes\mathbf{E} + 2G^*(\omega)\left(\Delta - \frac{1}{3}\mathbf{E}\otimes\mathbf{E}\right),
\tag{20}
$$

where $G^*(\omega) = G - K_G^*(\omega)$ is the complex shear modulus.

For an isotropic material, components of the tensor ${}^4\mathbf{C}^*(\omega)$ are obtained as follows:

$$
\begin{aligned}
C_{1111}^*(\omega) &= C_{2222}^*(\omega) = C_{3333}^*(\omega) = K + \frac{4}{3}G^*,\\
C_{1122}^*(\omega) &= C_{1133}^*(\omega) = C_{2233}^*(\omega) = K - \frac{2}{3}G^*,\\
C_{2323}^*(\omega) &= C_{1313}^*(\omega) = C_{1212}^*(\omega) = G^*.
\end{aligned}
\tag{20}
$$

5 Spectral relaxation and creep functions and kernels for a transversely isotropic material

Consider the spectral representation (13) for the transversally isotropic symmetry class. In this case $m=2$, $n=4$, $\mathbf{a}_{(1)}=\mathbf{e}_1\otimes\mathbf{e}_1$, $a_{(1)}=1$, $\mathbf{a}_{(2)}=\mathbf{E}-\mathbf{e}_1^2$, $a_{(2)}=\sqrt{2}$, ${}^4\boldsymbol{\Gamma}_{(3)}=\boldsymbol{\Delta}-\frac{1}{2}\left(\mathbf{E}-\mathbf{e}_1\otimes\mathbf{e}_1\right)\otimes\left(\mathbf{E}-\mathbf{e}_1\otimes\mathbf{e}_1\right)-\mathbf{e}_1\otimes\mathbf{e}_1\otimes\mathbf{e}_1\otimes\mathbf{e}_1-\frac{1}{2}\left(\mathbf{O}_2\otimes\mathbf{O}_2+\mathbf{O}_3\otimes\mathbf{O}_3\right)$, ${}^4\boldsymbol{\Gamma}_{(4)}=\frac{1}{2}\left(\mathbf{O}_2\otimes\mathbf{O}_2+\mathbf{O}_3\otimes\mathbf{O}_3\right)$, $\mathbf{O}_2=\mathbf{e}_3\otimes\mathbf{e}_1+\mathbf{e}_1\otimes\mathbf{e}_3$, $\mathbf{O}_3=\mathbf{e}_1\otimes\mathbf{e}_2+\mathbf{e}_2\otimes\mathbf{e}_1$ [21].

It should be noted that in this work it is assumed that the axis of transversal isotropy is codirectional with the basis vector $\mathbf{e}_1$.

Thus, for a transversally isotropic symmetry class, the spectral expansion of the tensor of relaxation functions has the form

$$
\begin{aligned}
{}^4\mathbf{R}(t)=R_{11}(t)\mathbf{e}_1^2\otimes\mathbf{e}_1^2+R_{12}(t)\frac{\mathbf{e}_1^2\otimes(\mathbf{E}-\mathbf{e}_1^2)+(\mathbf{E}-\mathbf{e}_1^2)\otimes\mathbf{e}_1^2}{\sqrt{2}}+R_{22}(t)\frac{\mathbf{E}-\mathbf{e}_1^2\otimes\mathbf{E}-\mathbf{e}_1^2}{2}+\\
+R_{33}(t)\left(\boldsymbol{\Delta}-\frac{1}{2}\left(\mathbf{E}-\mathbf{e}_1^2\right)\otimes\left(\mathbf{E}-\mathbf{e}_1^2\right)-\mathbf{e}_1^2\otimes\mathbf{e}_1^2-\frac{1}{2}\left(\mathbf{O}_2\otimes\mathbf{O}_2+\mathbf{O}_3\otimes\mathbf{O}_3\right)\right)+\\
+R_{44}(t)\frac{1}{2}\left(\mathbf{O}_2\otimes\mathbf{O}_2+\mathbf{O}_3\otimes\mathbf{O}_3\right).
\end{aligned}
$$

For tensors ${}^4\boldsymbol{\Pi}(t)$, ${}^4\mathbf{K}(t)$ and ${}^4\boldsymbol{\Gamma}(t)$ spectral representations will be similar to representation (21).

6 Exponential relaxation and creep functions and kernels

In practical problems, it is convenient to approximate the spectral functions of relaxation and creep as a sum of exponentials:

$$
\begin{aligned}
R_{\alpha\beta}(t)&=C_{\alpha\beta}-\sum_{\gamma=1}^{N_{\alpha\beta}}A_{\alpha\beta}^{(\gamma)}\left(1-\exp\left(-\frac{t}{\tau_{\alpha\beta}^{(\gamma)}}\right)\right),\\
\Pi_{\alpha\beta}(t)&=\Pi_{\alpha\beta}+\sum_{\gamma=1}^{n_{\alpha\beta}}B_{\alpha\beta}^{(\gamma)}\left(1-\exp\left(-\frac{t}{\psi_{\alpha\beta}^{(\gamma)}}\right)\right).
\end{aligned}
\tag{21}
$$

Here $A_{\alpha\beta}^{(\gamma)}$ and $\tau_{\alpha\beta}^{(\gamma)}$ are constants called spectra of relaxation values and relaxation times, while $B_{\alpha\beta}^{(\gamma)}$ and $\psi_{\alpha\beta}^{(\gamma)}$ are called spectra of creep values and creep times respectively. The quantities $C_{\alpha\beta}$ and $\Pi_{\alpha\beta}$ are constants in a similar spectral expansion for the tensors ${}^4\mathbf{C}$ and ${}^4\boldsymbol{\Pi}$.

It is important to note that such a representation as a sum of exponentials is written specifically for the spectral functions and creep and relaxation kernels (or functions that differ from them by a coefficient), and not for the components $R_{ijkl}(t)$, $\Pi_{ijkl}(t)$ $K_{ijkl}(t)$ and $\Gamma_{ijkl}(t)$.

If relations (6) and (10) are represented as the sum of the imaginary and real parts

$$\begin{aligned} {}^4\mathbf{C}^*(\omega) &= {}^4\mathbf{C}' + i\,{}^4\mathbf{C}'', \\ {}^4\mathbf{\Pi}^*(\omega) &= {}^4\mathbf{\Pi}' - i\,{}^4\mathbf{\Pi}'', \end{aligned} \tag{22}$$

then, by performing a direct Fourier transform from (22), taking into account (14), we obtain formulas for calculating the complex tensors of the elastic moduli and elastic compliances.

$$\begin{aligned} C'_{\alpha\beta} &= C_{\alpha\beta} - \sum_{\gamma=1}^{n} \frac{A_{\alpha\beta}^{(\gamma)}}{1+\left(\omega\tau_{\alpha\beta}^{(\gamma)}\right)^2}, \qquad C''_{\alpha\beta} = \sum_{\gamma=1}^{n} \frac{A_{\alpha\beta}^{(\gamma)}\omega\tau_{\alpha\beta}^{(\gamma)}}{1+\left(\omega\tau_{\alpha\beta}^{(\gamma)}\right)^2}, \\ \Pi'_{\alpha\beta} &= \Pi_{\alpha\beta} + \sum_{\gamma=1}^{n} \frac{B_{\alpha\beta}^{(\gamma)}}{1+\left(\omega\psi_{\alpha\beta}^{(\gamma)}\right)^2}, \qquad \Pi''_{\alpha\beta} = \sum_{\gamma=1}^{n} \frac{B_{\alpha\beta}^{(\gamma)}\omega\psi_{\alpha\beta}^{(\gamma)}}{1+\left(\omega\psi_{\alpha\beta}^{(\gamma)}\right)^2}. \end{aligned} \tag{23}$$

7 Problem statement in periodicity cell (PC) of the composite in Fourier transforms

In order to calculate the effective viscoelastic characteristics of fibers represented as a 1D composite under harmonic loading, we use the asymptotic averaging method [16, 25]. Using a variant of this method proposed in [13], to achieve this goal, it is necessary to consider a series of local problems L_{pq} on the 1/8th periodicity cell (PC) $\tilde{V}_\xi = \{\xi_i \mid 0 < \xi_i < a_i\}$ in dimensionless local coordinates ξ_i under harmonic loading:

$$\begin{cases} \nabla_\xi \cdot \boldsymbol{\sigma}^*_{(pq)} = \mathbf{0}, \ \text{в } \tilde{V}_\xi \\ \boldsymbol{\varepsilon}^*_{(pq)} = \dfrac{1}{2}\left(\nabla_\xi \otimes \mathbf{u}^*_{(pq)} + \left(\nabla_\xi \otimes \mathbf{u}^*_{(pq)}\right)^T\right), \ \text{в } \tilde{V}_\xi \\ \boldsymbol{\sigma}^*_{(pq)} = {}^4\mathbf{C}^* \cdot\cdot\, \boldsymbol{\varepsilon}^*_{(pq)}, \ \text{в } \tilde{V}_\xi \cup \Sigma'_s \cup \Sigma_s \\ \left[\mathbf{u}^*_{(pq)}\right] = \mathbf{0}, \ \ \mathbf{n} \cdot \left[\boldsymbol{\sigma}^*_{(pq)}\right] = \mathbf{0}, \ \text{on } \Sigma_{\xi\alpha N} \end{cases} \tag{24}$$

where a_i are the lengths of the edges of the 1/8 PC along the coordinate directions, $\Sigma_s = \{\xi_s = 0\}$ – coordinate planes, $\Sigma'_s = \{\xi_s = a_s\}$ – end planes of PC, $\Sigma_{\xi\alpha N}$ –

component interface planes in PC, $\nabla_\xi = \mathbf{e}_s \partial/\partial\xi_s$ – nabla operator relative to local coordinates ξ_i, $\mathbf{e}_s$ – vectors of a Cartesian (orthonormal) basis oriented along the PC edges. In (18), $\boldsymbol{\sigma}^*_{(pq)}, \boldsymbol{\varepsilon}^*_{(pq)}, \mathbf{u}^*_{(pq)}$ are the tensors of complex amplitudes of stresses, strains, and displacement vectors in the PC in the problem L_{pq}.

Problems (25) are supplemented with boundary conditions on surfaces Σ_s and Σ':

$$\begin{aligned} &\text{if } p=q: \mathbf{u}^*_{(pp)} \cdot \mathbf{e}_s = \frac{1}{2} \vartheta_s \overline{\varepsilon}^*_{pp} \delta_{sp}, \ \ \mathbf{e}_s \cdot \boldsymbol{\sigma}^*_{(pp)} \cdot \mathbf{e}_h = 0, \ \ \mathbf{e}_s \cdot \boldsymbol{\sigma}^*_{(pp)} \cdot \mathbf{e}_r = 0, \\ &\qquad s \neq h \neq r \neq s, \\ &\text{if } p \neq q: \mathbf{u}^*_{(pq)} \cdot \mathbf{e}_h = \frac{1}{4} \vartheta_s \overline{\varepsilon}^*_{pq} \delta_{sp}, \ \ \mathbf{e}_s \cdot \boldsymbol{\sigma}^*_{(pq)} \cdot \mathbf{e}_s = 0, \ \ \mathbf{u}^*_{(pq)} \cdot \mathbf{e}_r = 0, \\ &\qquad s \neq h \neq r \neq s, \end{aligned} \tag{25}$$

here $\vartheta_s = 0$ on Σ_s and $\vartheta_s = 1$ on Σ'_s, and $\overline{\varepsilon}^*_{pq}$ – specified components of the complex amplitudes tensor of average strains of the composite in the basis $\mathbf{e}_s$. There is no summation over dumb indices p and q here and below.

For the numerical solution of local problems (25) with BC (26), the finite element method is used. The software implementation of the solution of these problems was carried out in the SMCM software package, developed at the Department of Computational Mathematics and Mathematical Physics and the REC "Supercomputer Engineering Modeling and Development of Software Systems" of Moscow State Technical University. N.E. Bauman [26].

8 Calculation of effective complex kernels for the 1D composite material

After solving local problems (25) with BC (26), the components of the effective complex elastic moduli tensor ${}^4\mathbf{C}^{*1D}(\omega)$ of the 1D composite can be obtained by formula

$$\overline{C}^{*1D}_{ijpq}(\omega) = \frac{\left\langle \sigma^*_{ij(pq)} \right\rangle}{\overline{\varepsilon}^*_{pq}}, \tag{26}$$

here are the average stresses over PC $\tilde{V}_\xi$ are denoted

$$\left\langle \sigma^*_{ij(pq)} \right\rangle = 8 \int\limits_{\tilde{V}_\xi} \sigma^*_{ij(pq)} d\tilde{V}_\xi. \tag{27}$$

The effective complex compliance tensor ${}^4\bar{\mathbf{\Pi}}^{*1D}(\omega)$ of the composite is calculated as a tensor inverse to ${}^4\bar{\mathbf{C}}^{*1D}(\omega)$, i.e. satisfying

$$ {}^4\bar{\mathbf{\Pi}}^{*1D}(\omega)\cdot\cdot{}^4\bar{\mathbf{C}}^{*1D}(\omega)=\mathbf{\Delta}. \tag{28}$$

Then the effective tensors of the relaxation kernels and creep kernels of the composite material are found using the formulas

$$ \begin{aligned} {}^4\bar{\mathbf{K}}^{*1D}(\omega) &= {}^4\bar{\mathbf{C}}^{1D}-{}^4\bar{\mathbf{C}}^{*1D}(\omega),\\ {}^4\bar{\mathbf{\Gamma}}^{*1D}(\omega) &= {}^4\bar{\mathbf{\Pi}}^{*1D}(\omega)-{}^4\bar{\mathbf{\Pi}}^{1D}. \end{aligned} \tag{29}$$

In our case, the PC of a 1D composite is transversely isotropic with the main axis $O\mathbf{e}_1$, therefore, the tensors ${}^4\bar{\mathbf{K}}^{*1D}(\omega)$ and ${}^4\bar{\mathbf{\Gamma}}^{*1D}(\omega)$ are indifferent with respect to this symmetry group and can be represented in the form (21). Then, using the components obtained during the calculation in the basis $O\mathbf{e}_s$, we obtain the spectral complex constants $\bar{K}^{*1D}_{\alpha\beta}(\omega)$ и $\bar{\Gamma}^{*1D}_{\alpha\beta}(\omega)$. If these components are presented in the form

$$ \begin{aligned} \bar{K}^{*1D}_{\alpha\beta}(\omega) &= \bar{K}^{1D\prime}_{\alpha\beta}(\omega)-i\bar{K}^{1D\prime\prime}_{\alpha\beta}(\omega),\\ \bar{\Gamma}^{*1D}_{\alpha\beta}(\omega) &= \bar{\Gamma}^{1D\prime}_{\alpha\beta}(\omega)-i\bar{\Gamma}^{1D\prime\prime}_{\alpha\beta}(\omega), \end{aligned} \tag{30}$$

then the functions in the time domain can be formally obtained by applying the inverse cosine and sine Fourier transforms

$$ \begin{aligned} \bar{K}^{1D}_{\alpha\beta}(t) &= \frac{1}{\pi}\int_0^\infty \bar{K}^{1D\prime}_{\alpha\beta}(\omega)\cos(\omega t)\,d\omega+\frac{1}{\pi}\int_0^\infty \bar{K}^{1D\prime\prime}_{\alpha\beta}(\omega)\sin(\omega t)\,d\omega,\\ \bar{\Gamma}^{1D}_{\alpha\beta}(t) &= \frac{1}{\pi}\int_0^\infty \bar{\Gamma}^{1D\prime}_{\alpha\beta}(\omega)\cos(\omega t)\,d\omega+\frac{1}{\pi}\int_0^\infty \bar{\Gamma}^{1D\prime\prime}_{\alpha\beta}(\omega)\sin(\omega t)\,d\omega. \end{aligned} \tag{31}$$

9 Approximation of complex kernels for 1D composite material

The main goal of this section is to obtain spectral kernels in the form of a sum of exponents

$$\bar{K}_{\alpha\beta}^{Approx}(t)=\sum_{\gamma=1}^{N}\frac{\bar{A}_{\alpha\beta}^{1D(\gamma)}}{\bar{\tau}_{\alpha\beta}^{1D(\gamma)}}\exp\left(-\frac{t}{\bar{\tau}_{\alpha\beta}^{1D(\gamma)}}\right),$$
$$\bar{\Gamma}_{\alpha\beta}^{Approx}(t)=\sum_{\gamma=1}^{N}\frac{\bar{B}_{\alpha\beta}^{1D(\gamma)}}{\bar{\psi}_{\alpha\beta}^{1D(\gamma)}}\exp\left(-\frac{t}{\bar{\psi}_{\alpha\beta}^{1D(\gamma)}}\right). \quad (32)$$

The calculation of integrals in (32) leads to a significant accumulation of a numerical error, which is expressed in the oscillation of the obtained functions. Therefore, direct approximation of kernels $\bar{K}_{\alpha\beta}^{1D}(t)$ and $\bar{\Gamma}_{\alpha\beta}^{1D}(t)$ by functions (33) will be extremely difficult. A more rational idea in this case would be the approximation of complex spectral kernels $\bar{K}_{\alpha\beta}^{*1D}(\omega)$ and $\bar{\Gamma}_{\alpha\beta}^{*1D}(\omega)$ by functions of the form

$$\bar{K}_{\alpha\beta}^{*Approx}(\omega)=\sum_{\gamma=1}^{N_{\alpha\beta}}\frac{\bar{A}_{\alpha\beta}^{1D(\gamma)}}{1+\left(\omega\bar{\tau}_{\alpha\beta}^{1D(\gamma)}\right)^2}-i\sum_{\gamma=1}^{n}\frac{\bar{A}_{\alpha\beta}^{1D(\gamma)}\omega\tau_{\alpha\beta}^{1D(\gamma)}}{1+\left(\omega\bar{\tau}_{\alpha\beta}^{1D(\gamma)}\right)^2},$$
$$\bar{\Gamma}_{\alpha\beta}^{*Approx}(\omega)=\sum_{\gamma=1}^{n_{\alpha\beta}}\frac{\bar{B}_{\alpha\beta}^{1D(\gamma)}}{1+\left(\omega\bar{\psi}_{\alpha\beta}^{1D(\gamma)}\right)^2}-i\sum_{\gamma=1}^{n}\frac{\bar{B}_{\alpha\beta}^{1D(\gamma)}\omega\bar{\psi}_{\alpha\beta}^{1D(\gamma)}}{1+\left(\omega\bar{\psi}_{\alpha\beta}^{1D(\gamma)}\right)^2}. \quad (33)$$

The approximation of kernels (31) by functions (34) is equivalent to the approximation of kernels (32) by functions (33), but at the same time it allows avoiding the numerical error in calculating the integrals in (32).

For a qualitative approximation in the form (34), the effective complex kernels $\bar{K}_{\alpha\beta}^{*1D}(\omega)$ and $\bar{\Gamma}_{\alpha\beta}^{*1D}(\omega)$ are calculated for a certain frequency spectrum ω_k, $k=1...N$, and the better the chosen frequency spectrum reflects all the features of the functions $\bar{K}_{\alpha\beta}^{*1D}(\omega)$ and $\bar{\Gamma}_{\alpha\beta}^{*1D}(\omega)$, the better the approximation can be obtained.

To obtain the material constants $\bar{A}_{\alpha\beta}^{1D(\gamma)}$, $\bar{\tau}_{\alpha\beta}^{1D(\gamma)}$, $\bar{B}_{\alpha\beta}^{1D(\gamma)}$ and $\bar{\psi}_{\alpha\beta}^{1D(\gamma)}$, the problem of minimizing the functionals

$$\Omega_K(\bar{A}_{\alpha\beta}^{1D(\gamma)},\bar{\tau}_{\alpha\beta}^{1D(\gamma)})=\sum_{k=1}^{N}\left(\bar{K}_{\alpha\beta}^{1D\prime}(\omega_k)-\sum_{\gamma=1}^{N_{\alpha\beta}}\frac{\bar{A}_{\alpha\beta}^{1D(\gamma)}}{1+\left(\omega_k\bar{\tau}_{\alpha\beta}^{1D(\gamma)}\right)^2}\right)^2+\sum_{k=1}^{N}\left(\bar{K}_{\alpha\beta}^{1D\prime\prime}(\omega_k)-\sum_{\gamma=1}^{N_{\alpha\beta}}\frac{\bar{A}_{\alpha\beta}^{1D(\gamma)}\omega_k\tau_{\alpha\beta}^{1D(\gamma)}}{1+\left(\omega_k\bar{\tau}_{\alpha\beta}^{1D(\gamma)}\right)^2}\right)^2, \quad (35)$$
$$\Omega_\Gamma(\bar{B}_{\alpha\beta}^{1D(\gamma)},\bar{\psi}_{\alpha\beta}^{1D(\gamma)})=\sum_{k=1}^{N}\left(\bar{\Gamma}_{\alpha\beta}^{1D\prime}(\omega_k)-\sum_{\gamma=1}^{n_{\alpha\beta}}\frac{\bar{B}_{\alpha\beta}^{1D(\gamma)}}{1+\left(\omega_k\bar{\psi}_{\alpha\beta}^{1D(\gamma)}\right)^2}\right)^2+\sum_{k=1}^{N}\left(\bar{\Gamma}_{\alpha\beta}^{1D\prime\prime}(\omega_k)-\sum_{\gamma=1}^{n_{\alpha\beta}}\frac{\bar{B}_{\alpha\beta}^{1D(\gamma)}\omega_k\bar{\psi}_{\alpha\beta}^{1D(\gamma)}}{1+\left(\omega_k\bar{\psi}_{\alpha\beta}^{1D(\gamma)}\right)^2}\right)^2.$$

The spectra of values $\bar{A}_{\alpha\beta}^{1D(\gamma)}$ and $\bar{B}_{\alpha\beta}^{1D(\gamma)}$ are calculated from the spectra of times using the least squares method. As a result, we obtain 5 independent spectral exponential kernels for a 1D composite.

10 Numerical simulation results for 1D glass/epoxy composite

Consider a 1D composite consisting of an epoxy matrix and glass fibers.

In numerical calculations, the following values of the elastic characteristics of the matrix (modulus of elasticity E_m and Poisson ratio ν_m), as well as fibers (modulus of elasticity E_f and Poisson ratio ν_f) were used:

$$E_m = 3.3 \text{ GPa}, \ \nu_m = 0.35, \ E_f = 200 \text{ GPa}, \ \nu_f = 0.25.$$

The shear creep kernel the matrix is also specified:

$$K_G(t) = \sum_{\gamma=1}^{N} \frac{A_G^{(\gamma)}}{\tau_G^{(\gamma)}} \exp\left(-\frac{t}{\tau_G^{(\gamma)}} \right),$$

where $N = 3$, $A_G^{(1)} = 0.354$ GPa, $A_G^{(2)} = 0.483$ GPa, $A_G^{(3)} = 0.2175$ GPa, $\tau_G^{(1)} = 4.218 \cdot 10^3$ s, $\tau_G^{(2)} = 5.793 \cdot 10^6$ s, $\tau_G^{(3)} = 7.792 \cdot 10^7$ s. The fibers are assumed to be elastic.

For a 1D composite, the following spectra of values and times of relaxation were obtained:

$$\overline{A}_{11}^{1D(1)} = 0.197 \text{ GPa}, \ \overline{A}_{11}^{1D(2)} = 0.27 \text{ GPa}, \ \overline{A}_{11}^{1D(3)} = 0.122 \text{ GPa},$$

$$\overline{\tau}_{11}^{1D(1)} = 4.224 \cdot 10^3 \text{ s}, \ \overline{\tau}_{11}^{1D(2)} = 5.81 \cdot 10^6 \text{ s}, \ \overline{\tau}_{11}^{1D(3)} = 7.8 \cdot 10^7 \text{ s},$$

$$\overline{A}_{12}^{1D(1)} = 0.0793 \text{ GPa}, \ \overline{A}_{12}^{1D(2)} = 0.113 \text{ GPa}, \ \overline{A}_{12}^{1D(3)} = 0.0532 \text{ GPa},$$

$$\overline{\tau}_{12}^{1D(1)} = 4.292 \cdot 10^3 \text{ s}, \ \overline{\tau}_{12}^{1D(2)} = 5.94 \cdot 10^6 \text{ s}, \ \overline{\tau}_{12}^{1D(3)} = 7.9 \cdot 10^7 \text{ s},$$

$$\overline{A}_{22}^{1D(1)} = 1.37 \text{ GPa}, \ \overline{A}_{22}^{1D(2)} = 1.89 \text{ GPa}, \ \overline{A}_{22}^{1D(3)} = 0.87 \text{ GPa},$$

$$\overline{\tau}_{22}^{1D(1)} = 4.245 \cdot 10^3 \text{ s}, \ \overline{\tau}_{22}^{1D(2)} = 5.84 \cdot 10^6 \text{ s}, \ \overline{\tau}_{22}^{1D(3)} = 7.83 \cdot 10^7 \text{ s},$$

$$\overline{A}_{33}^{1D(1)} = 1.698 \text{ GPa}, \ \overline{A}_{33}^{1D(2)} = 0.3 \text{ GPa}, \ \overline{A}_{33}^{1D(3)} = 2.38 \text{ GPa},$$

$$\overline{A}_{33}^{1D(4)} = 0.68 \text{ GPa}, \ \overline{A}_{33}^{1D(5)} = 1.195 \text{ GPa}, \ \overline{A}_{33}^{1D(6)} = 0.534 \text{ GPa},$$

$$\overline{\tau}_{33}^{1D(1)} = 4.253 \cdot 10^3 \text{ s}, \ \overline{\tau}_{33}^{1D(2)} = 5.186 \cdot 10^3 \text{ s}, \ \overline{\tau}_{33}^{1D(3)} = 5.865 \cdot 10^6 \text{ s},$$

$$\overline{\tau}_{33}^{1D(4)} = 8.428 \cdot 10^{6} \text{ s}, \ \overline{\tau}_{33}^{1D(5)} = 7.911 \cdot 10^{7} \text{ s}, \ \overline{\tau}_{33}^{1D(6)} = 1.021 \cdot 10^{8} \text{ s},$$

$$\overline{A}_{44}^{1D(1)} = 2.451 \text{ GPa}, \ \overline{A}_{44}^{1D(2)} = 3.459 \text{ GPa}, \ \overline{A}_{44}^{1D(3)} = 1.614 \text{ GPa},$$

$$\overline{\tau}_{44}^{1D(1)} = 4.282 \cdot 10^{3} \text{ s}, \ \overline{\tau}_{44}^{1D(2)} = 5.915 \cdot 10^{6} \text{ s}, \ \overline{\tau}_{44}^{1D(3)} = 8.21 \cdot 10^{7} \text{ s},$$

Figures 1-5 show the effective spectral relaxation functions $\overline{R}_{11}^{1D}(t)$, $\overline{R}_{12}^{1D}(t)$, $\overline{R}_{22}^{1D}(t)$, $\overline{R}_{33}^{1D}(t)$ and $\overline{R}_{44}^{1D}(t)$ on a semi-logarithmic time scale obtained using the developed method.

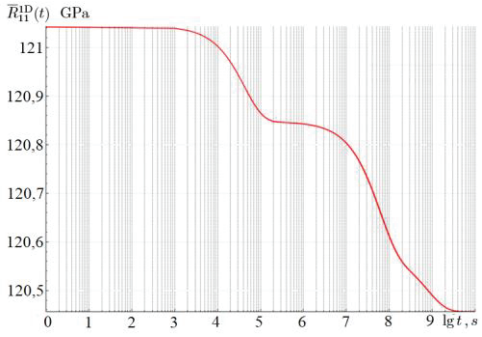

Fig. 1. Effective relaxation function $\overline{R}_{11}^{1D}(t)$ for 1D glass/epoxy composite.

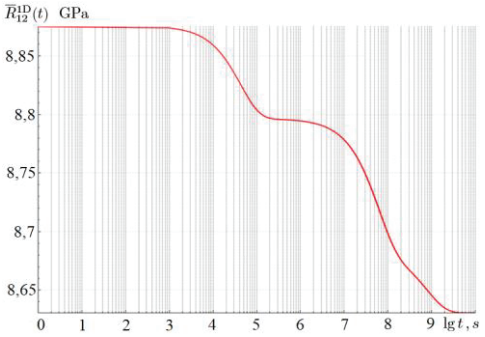

Fig. 2. Effective relaxation function $\overline{R}_{12}^{1D}(t)$ for 1D glass/epoxy composite.

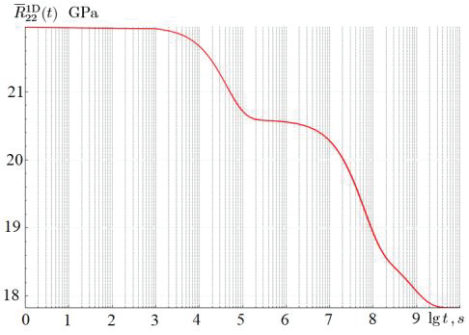

Fig. 3. Effective relaxation function $\overline{R}_{22}^{1D}(t)$ for 1D glass/epoxy composite.

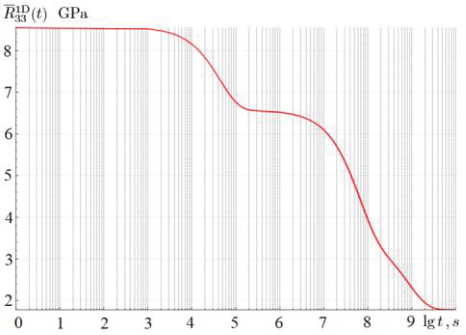

Fig. 4. Effective relaxation function $\overline{R}_{33}^{1D}(t)$ for 1D glass/epoxy composite.

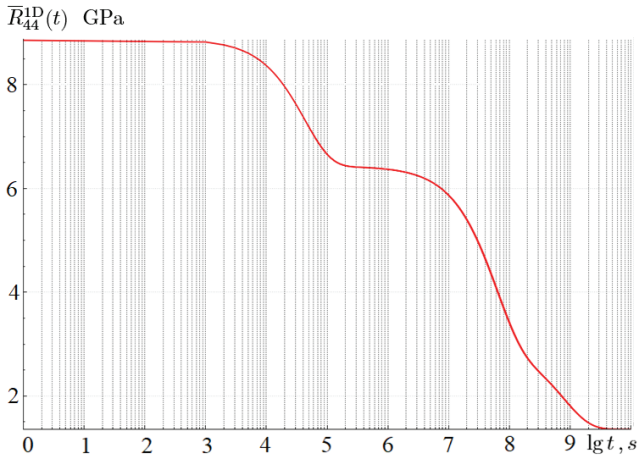

Fig. 5. Effective relaxation function $\overline{R}_{44}^{1D}(t)$ for 1D glass/epoxy composite.

11 Conclusions

A method for calculating the effective integral viscoelastic characteristics of unidirectional composites has been developed. A method is proposed for approximating the Fourier transforms of effective kernels, which makes it possible to avoid the numerical error of integration in the process of the inverse Fourier transform. Numerical modeling of 1D fiberglass is carried out, showing the consistency of the proposed method for calculating effective kernels and the approximation method.

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