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Modeling the edge effect in composites based on asymptotic homogenization method

Yu. Dimitrienko[1*], *Yu.* Yurin[1], and *D.* Chebakov[1]

[1]Bauman Moscow State Technical University 5, 2-ya Baumanskaya str., 105005 Moscow, Russian Federation

Abstract. The purpose of this work is to develop a numerical algorithm for solving the problem of calculating the stress-strain state of structures made of composite materials, taking into account edge effects. The solution of the problem is carried out by modifying the asymptotic averaging method by taking into account an additional solution of the boundary layer type, which quickly fades away from the boundary. This solution makes a significant contribution to the neighborhood of the composite surface itself. A numerical example of calculating an element of a composite structure with holes is given, which showed the feasibility of the proposed method. The calculations are compared with a three-dimensional finite element solution in the ANSYS software package. **Key words:** composite materials, edge effects, numerical algorithm, the stress-strain state.

1 Introduction

Existing research in the theory of composite materials is based on the homogenization approach, which consists in replacing a heterogeneous medium with a homogeneous one with certain effective properties. These effective properties are derived from internal asymptotics, which excludes boundary layer effects. However, at the boundary of the construction, the periodic solution is violated. This loss of periodicity of the internal asymptotic solution must be taken into account.

From a mathematical point of view, homogenization theory is a limiting theory that uses asymptotic expansion and the assumption of periodicity to replace differential equations with rapidly oscillating coefficients [1-3]. In elementary cell models, global properties are determined taking into account the conditions of macroscopic periodicity. Local problems are solved for specific representative load cases, thus representing the actual interaction between macro and micro-scale deformations.

Currently, there are some approaches to modeling edge effects in the literature [1,4], however, as a rule, model problems are considered, without taking into account the real curvilinear shape of the surface of structures. The purpose of this work is to develop a variant of the asymptotic averaging method, which allows calculations of the stress-strain state of

* Corresponding author: dimit.bmstu@gmail.com

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composite structures taking into account the edge effect with an arbitrary shape of the surface of the structure.

2 Asymptotic solutions of the problem of elasticity theory taking into account the boundary effect for a near-periodic composite

Consider a composite V with a periodic structure, the outer boundary $\Sigma = \Sigma_\sigma + \Sigma_u$ of which has the equation

$$\Sigma :, \ f(x^i) = 0. \tag{1}$$

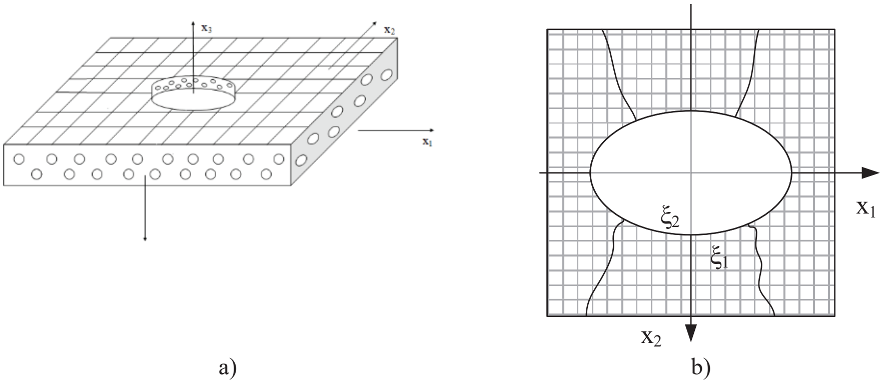
Taking into account the quasi-periodicity of the composite [5,6], the surface equation can be considered as functions of two variables $\overline{x}^i$ and ξ^i :

$$f(\overline{x}^i, \xi^j) = 0 \tag{2}$$

where x_i is the Cartesian coordinates for the characteristic size of the structure and the local system ξ_i - is the "fast" (local) coordinates for the representative volume element (RVE):

$$\overline{x}_i = \frac{x_i}{L}, \qquad \xi_i = \frac{\overline{x}_i}{\kappa} = \frac{x_i}{\ell} \tag{3}$$

Here $\kappa = l / L$ is a small parameter, L is the global scale of the length L of the area V of the entire composite, *l* is the local scale of the length.



a) b)

Fig. 1. A geometric model of a composite for constructing a solution taking into account the edge effect: a general composite model (a); areas V_α, where the frequency of functions $\hat{u}_k^{(n)}$ and the direction of attenuation of the solution for the boundary layer problem (b) change.

Let us consider the problem of linear elasticity theory [7,8] in the domain V with a boundary Σ. he solution of this problem of linear elasticity theory for the composite materials according to the method of asymptotic averaging [5,9,10], is sought in the form

$$u_k = u_k^{(0)}(\overline{x}^i) + \sum_{n=1}^{\infty} \kappa^n \tilde{u}_k^{(n)}, \tag{4}$$

where u_k are the components of the displacement vector, and

$$\tilde{u}_k^{(n)} = u_k^{(n)}(\overline{x}_i, \xi_j) + \hat{u}_k^{(n)}(\overline{x}_i, \xi_j), \;\; n > 0, \qquad \hat{u}_k^{(0)} \equiv 0 \; . \tag{5}$$

Functions $u_k^{(n)}(\overline{x}_i, \xi_j)$ are 1-periodic in all ξ_j 3 coordinates: $u_k^{(n)}(\overline{x}_i, \xi_j + 1) = u_k^{(n)}(\overline{x}_i, \xi_j)$, as result, the domain of definition of these functions are: $\xi_i \in V_\xi$, $\overline{x}_i \in \overline{V}$. Here V_ξ is one complete periodicity cell, and $\overline{V}$ is the average are of the composite material structure, its equation $\overline{V} = \left\{ \overline{x}_i : f(\overline{x}^i, \xi_j^*) < 0 \right\}$, where ξ_j^* is some fixed set of values of local coordinates corresponding to the surface Σ, for example $\xi_j^* = 0$

The functions $\hat{u}_k^{(n)}(\overline{x}_i, \overline{\xi}_j)$ in (5) are 1-periodic in two coordinates: ξ_3, ξ_α, and decaying in the third coordinate ξ_β, where $\alpha, \beta = 1, 2$, $\alpha \neq \beta$. The area V is divided into subdomain V_α, where: 1) $\hat{u}_k^{(n)}$ are periodic by ξ_1, ξ_3, and 2) $\hat{u}_k^{(n)}$ by ξ_2, ξ_3

The attenuation of functions $\hat{u}_k^{(n)}(x_i, \xi_j)$ by coordinate ξ_β, when moving away from the surface Σ is understood as follows:

$$\left| \hat{u}_k^{(n)} \right| \to 0, \quad \left| \xi_\beta \right| \to \infty \, . \tag{6}$$

Based on formula (4), we obtain an asymptotic expansion for deformations:

$$\varepsilon_{ij} = \sum_{n=0}^{\infty} \kappa^n \tilde{\varepsilon}_{ij}^{(n)} \tag{7}$$

$$\tilde{\varepsilon}_{ij}^{(n)} = \varepsilon_{ij}^{(n)}(x_k, \xi_m) + \hat{\varepsilon}_{ij}^{(n)}(x_k, \xi_m), \quad n \geq 0,$$

where

$$\varepsilon_{ij}^{(n)} = \frac{1}{2}\left(u_{i,j}^{(n)} + u_{j,i}^{(n)} \right) + \frac{1}{2}\left(u_{i/j}^{(n+1)} + u_{j/i}^{(n+1)} \right),$$

$$\hat{\varepsilon}_{ij}^{(n)} = \frac{1}{2}\left(\hat{u}_{i,j}^{(n)} + \hat{u}_{j,i}^{(n)}\right) + \frac{1}{2}\left(\hat{u}_{i/j}^{(n+1)} + \hat{u}_{j/i}^{(n+1)}\right), \tag{8}$$

Here the derivatives are denoted by $u_{i,j}^{(n)} = \partial u_i^{(n)} / \partial x_j$ and $u_{i/j}^{(n)} = \partial u_i^{(n)} / \partial \xi_j$.

Using the generalized Hooke's law, we find the asymptotic expansion for stresses:

$$\sigma_{ij} = \sum_{n=0}^{\infty} \kappa^n \tilde{\sigma}_{ij}^{(n)}, \tag{9}$$

$$\tilde{\sigma}_{ij}^{(n)} = \sigma_{ij}^{(n)}(x_k, \xi_m) + \hat{\sigma}_{ij}^{(n)}(x_k, \xi_m), \quad n \geq 0$$

where

$$\sigma_{ij}^{(n)} = C_{ijkl}\varepsilon_{kl}^{(n)}, \qquad \hat{\sigma}_{ij}^{(n)} = C_{ijkl}\hat{\varepsilon}_{kl}^{(n)}, \tag{10}$$

Here $C_{ijkl}(\xi_k)$ are the components of the tensor of the elastic modules of the composite [11], which are functions of ξ_k.

3 Formulations of local and averaged problems

Substituting decomposition (9) into the equilibrium equation for the composite $\sum_\sigma$ and $\sum_u$ and the boundary conditions on part of the surface and, as well as into the conditions on the interface of the phases of the composite $\sum_S$: we obtain:

$$\frac{1}{\kappa}(\sigma_{ij/j}^{(0)} + \hat{\sigma}_{ij/j}^{(0)}) + \sum_{n=0}^{\infty} \kappa^n (\sigma_{ij,j}^{(n)} + \sigma_{ij/j}^{(n+1)} + \hat{\sigma}_{ij,j}^{(n)} + \hat{\sigma}_{ij/j}^{(n+1)}) = 0 \tag{11}$$

$$\Sigma_\sigma : \quad n_i\left(\sigma_{ij}^{(0)} + \hat{\sigma}_{ij}^{(0)} + \sum_{n=1}^{\infty}\kappa^n\left(\sigma_{ij}^{(n)} + \hat{\sigma}_{ij}^{(n)}\right)\right) = S_{ie}, \quad \Sigma_u : \quad u_i^{(0)} + \sum_{n=1}^{\infty}\kappa^n\left(u_i^{(n)} + \hat{u}_i^{(n)}\right) = u_{ie},$$

$$\Sigma_S : \quad n_i\left[\sigma_{ij}^{(0)} + \hat{\sigma}_{ij}^{(0)} + \sum_{n=1}^{\infty}\kappa^n\left(\sigma_{ij}^{(n)} + \hat{\sigma}_{ij}^{(n)}\right)\right] = 0, \quad \sum_{n=1}^{\infty}\kappa^n [u_i^{(n)} + \hat{u}_i^{(n)}] = 0,$$

- here $[u_i^{(n)}]$ is a jump of functions at the component interface.

From the system of asymptotic equations we obtain 2 types of recurrent sequences of local problems:

type 1 tasks:

$$\sigma_{ij/j}^{(n)} + \sigma_{ij,j}^{(n-1)} = h_i^{(n-1)}, \quad в\ V_\xi$$
$$\sigma_{ij}^{(n)} = C_{ijkl}\varepsilon_{kl}^{(n)} \qquad \varepsilon_{ij}^{(n)} = \frac{1}{2}\left(u_{i,j}^{(n)} + u_{j,i}^{(n)}\right) + \frac{1}{2}\left(u_{i/j}^{(n+1)} + u_{j/i}^{(n+1)}\right)$$
$$\sum_s : \quad n_i\left[\sigma_{ij}^{(n)}\right] = 0, \quad \left[u_i^{(n+1)}\right] = 0 \tag{12}$$
$$\left\langle u_i^{(n+1)} \right\rangle_{V_\xi} = 0, \quad n = 0,1,2...$$

and tasks of the 2nd type:

$$\hat{\sigma}_{ij/j}^{(n)} + \hat{\sigma}_{ij,j}^{(n-1)} = 0 \qquad в\ \hat{V}_\xi$$
$$\hat{\sigma}_{ij/j}^{(n)} = C_{ijkl}\hat{\varepsilon}_{kl}^{(n)}, \qquad \hat{\varepsilon}_{ij}^{(n)} = \frac{1}{2}\left(\hat{u}_{i,j}^{(n)} + \hat{u}_{j,i}^{(n)}\right) + \frac{1}{2}\left(\hat{u}_{i/j}^{(n+1)} + \hat{u}_{j/i}^{(n+1)}\right),$$
$$\sum_s : \quad n_i\left[\hat{\sigma}_{ij}^{(n)}\right], \quad [\hat{u}_i^{(n+1)}] = 0 \tag{13}$$
$$\sum_{\xi\sigma} : \quad n_i\hat{\sigma}_{ij}^{(n)} = -n_i\sigma_{ij}^{(n)} + S_{ie}^{(n)}$$
$$\hat{u}_i^{(n+1)} \to 0, \quad \left|\xi_\beta\right| \to \infty \ n = 0,1,2...$$

Here are indicated:

$$h_i^{(-1)} = 0 \qquad h_i^{(n)} = <\sigma_{ij}^{(n)}>_{,j}, \quad n \ge 0, \qquad S_{ie}^{(0)} = S_{ie}, \ S_{ie}^{(0)} = 0 \ \ n > 0, \tag{14}$$

where $<\sigma_{ij}^{(n-1)}> = \int_{V_\varsigma} \sigma_{ij}^{(n-1)} dV_\xi$ are averaging by RVE. In the task (13) $\hat{\varepsilon}_{ij}^{(0)} = \frac{1}{2}\left(\hat{u}_{i/j}^{(1)} + \hat{u}_{j/i}^{(1)}\right)$.

Tasks of type 1 (12) are classical tasks on the periodicity cell. To achieve an acceptable accuracy of the solution, it is sufficient to consider only problem (12) at n=0 - the zero approximation problem. Its solution is the functions $u_i^{(1)}$ and $\sigma_{ij}^{(0)}$, parametrically dependent on the average deformations $\bar{\varepsilon}_{ij} = \frac{1}{2}\left(u_{i,j}^{(0)} + u_{j,i}^{(0)}\right)$

$$u_i^{(1)} = \sum_{p,q=1}^{3} U_{ipq}\bar{\varepsilon}_{pq}, \quad \sigma_{ij}^{(0)} = \sum_{p,q=1}^{3} \sigma_{ij(pq)}\bar{\varepsilon}_{pq}, \tag{15}$$

where U_{ipq} and $\sigma_{ij(pq)}$ are some functions defined in the process of solving the problem on RVE.

To calculate displacements $u_i^{(0)}$ and deformations $\bar{\varepsilon}_{ij}$ we have the following averaged problem

$$\left\langle \sigma_{ij}^{(0)} \right\rangle_{,j} = 0$$
$$\left\langle \sigma_{ij}^{(0)} \right\rangle = \bar{C}_{ijkl}\bar{\varepsilon}_{kl}, \qquad \bar{\varepsilon}_{ij} = \frac{1}{2}\left(u_{i,j}^{(0)} + u_{j,i}^{(0)}\right), \tag{16}$$
$$\bar{\Sigma}_\sigma : \quad n_j \left\langle \sigma_{ij}^{(n)} \right\rangle = \left\langle S_{ie} \right\rangle, \quad \Sigma_u : \quad u^{(0)} = \left\langle u_{ie} \right\rangle$$

The tensor of effective elastic modulus of a composite material is calculated by the formula [5]

$$\bar{C}_{ijpq} = \frac{< \sigma_{ij(pq)} >}{\bar{\varepsilon}_{pq}} \ . \tag{17}$$

Type 2 problems (13) are problems on a semi-infinite domain $\hat{V}_\xi$, with periodicity conditions in 2 directions ξ_3, ξ_α, and infinite in coordinate ξ_β. To achieve an acceptable accuracy of the solution, it is sufficient to consider only problem (13) at n=0 - the zero approximation problem. Its solution is the functions $\hat{u}_i^{(1)}$ and $\hat{\sigma}_{ij}^{(0)}$, depending on the input data: functions $\hat{S}_{ie} = -n_i\sigma_{ij}^{(0)} + S_{ie}$, defined on a part of the surface $\sum_{\xi\sigma} = \sum_\sigma \bigcap V_\xi$.

The resulting stresses and deformations in the zero approximation have the form

$$\tilde{\sigma}_{ij} = \sigma_{ij}^{(0)} + \hat{\sigma}_{ij}^{(0)}, \quad \tilde{\varepsilon}_{ij} = \varepsilon_{ij}^{(0)} + \hat{\varepsilon}_{ij}^{(0)} .$$

4 Numerical simulation of microstresses and effective elastic characteristics of composites for a type 1 problem

The numerical implementation of the developed method was considered on the example of the problem of stretching the structure in the form of a joint of 2 composite plates with holes in the joint zone (Fig. 2). The case of a fabric composite was considered, the periodicity cell of which is shown in Fig.2b. The same figure shows the finite element grid used to solve the problem (12). The results of solving local problems on RVE were discussed in detail in [9], so we will not consider them in detail. The calculations considered a glass/epoxy composite.

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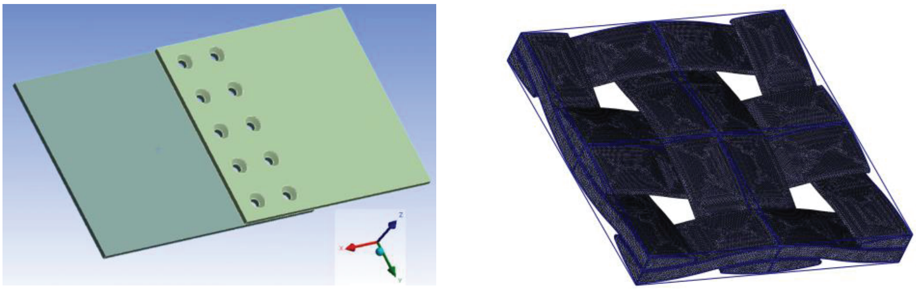

Fig. 2. Composite plate with holes (a) and finite element mesh for composite RVE (b), which were used in the calculations.

Tetrahedral finite elements with linear approximation were used in the calculations. The number of finite elements was 282266, the number of nodes was 46397.

Some results of solving the averaged problem (16) are shown in Fig.3. loading of the structure of 2 plates was carried out by longitudinal force, while the holes of the plates were supposed to be free from loading. A tetrahedral grid with linear approximation was used for calculations, the number of elements was 2,086,414. The number of holes varied during the calculations.

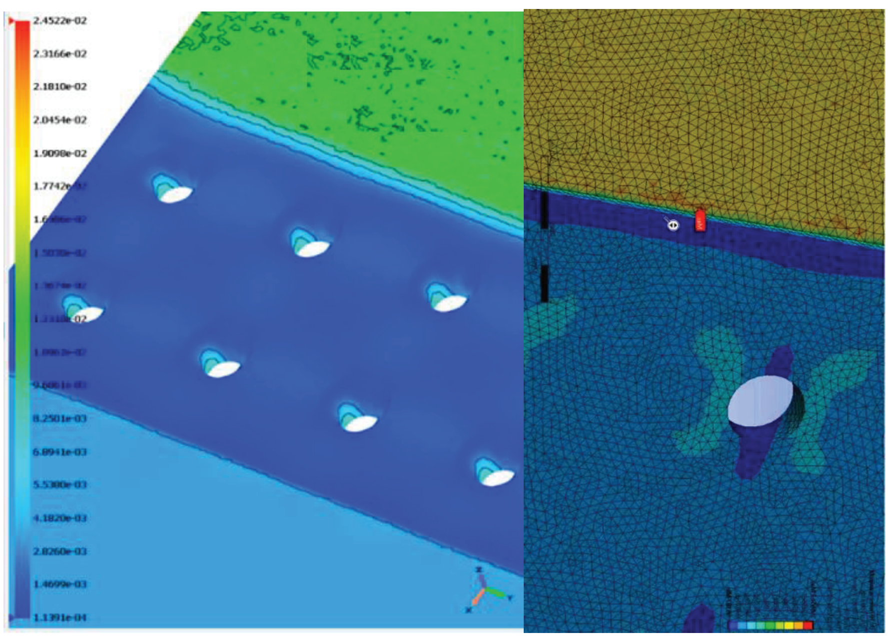

Fig. 3. Distribution of the stress field $\left\langle \sigma_{xx}^{(0)} \right\rangle$ in the composite plate, obtained using the SMCM PC (a) and ANSYS PC (b).

Finite element calculations were carried out using the SMCM software package developed at the Scientific and Educational Center "Supercomputer Engineering Modeling and Development of Software Complexes" of Bauman Moscow State Technical University [16,17] (SIMPLEX), as well as using the ANSYS software package. Comparative results of calculations for the maximum value of the normal stress along the longitudinal axis of tension OX $\left\langle \sigma_{xx}^{(0)} \right\rangle$ are shown in Fig. 2.

2 types of finite element grid were used for calculations: 300 thousand finite elements with 2 FE in plate thickness, and 15 million FE with 4 FE in plate thickness.

Table 1. Values of maximum stresses in the vicinity of holes, depending on their number.

Number of holes	$\max \left\langle \sigma_{xx}^{(0)} \right\rangle$ **Ansys,** (MPa)	$\max \left\langle \sigma_{xx}^{(0)} \right\rangle$ **SMCM,** (MPa)
4	25.76	25.73
6	26.46	26.39
8	27.05	27.09
10	28.52	28.43

It was found that the results of calculations using the ANSYS and SMCM PCs coincide quite well σ_{xx}, both on a coarse grid (300 thousand finite elements, 2 FE in thickness) and on a fine grid (4 FE in thickness, 15 million FE). In the zone of rigid pinching, a state of pure bending is realized when σ_{xx} the linear depends on the transverse coordinate of the plate, because of this, the maximum values $\max \sigma_{xx}$ are realized in the zone of load application at the ends of the plate. Local stress maxima $\max \left\langle \sigma_{xx}^{(0)} \right\rangle$ are also reached in the vicinity of the holes, the values of which are presented in Table 1.

Calculations have shown that with an increase in the number of holes in the connection, the stresses $\max \left\langle \sigma_{xx}^{(0)} \right\rangle$ increase.

5 Conclusions

A method for modeling the edge effect in composite materials based on the method of asymptotic averaging is proposed. 3 types of problems that arise in this method are formulated: problems on the periodicity cell, an averaged problem for a composite with effective characteristics, and an additional local problem with a decaying solution at a distance from the boundary of the body surface.

An example of numerical simulation of the stress-strain state of a composite composite structure with holes is given, which demonstrated the fundamental feasibility of the proposed calculation method. A comparison of the solution of the averaged problem of the stress-strain state of a composite plate obtained using the SMCM software package developed at Bauman Moscow State Technical University and using the ANSYS complex showed high accuracy of the numerical method and the software package.

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