Modeling of aeroelastic composite plates vibrations based on asymptotic theory

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Abstract. The paper is devoted to solving the problem of aeroelastic deformation of multilayer thin composite plates. A theory of aeroelastic deformations of composite plates based on the application of the asymptotic averaging method and piston theory for modeling pressure on the body surface is proposed. The averaged equations of aeroelastic vibrations of composite plates are derived. An example of a numerical solution of the problem of one-dimensional bending vibrations of a composite plate based on the developed theory is given. **Key words:** multilayer composite, aeroelastic deformations, asymptotic averaging method, numerical solution.

1 Introduction

Composite structures of modern aircraft, such as wings, rudders, are not completely rigid; when exposed to a high-speed flow, aeroelastic phenomena occur when structural deformations cause changes in aerodynamic forces. Additional aerodynamic forces cause an increase in structural deformations, which leads to an increase in aerodynamic forces. These interactions may decrease to a state of equilibrium or may diverge catastrophically. In this regard, preliminary aeroelastic modeling of the behavior of a structural element before the start of real tests is of particular importance in order to notice and eliminate design flaws at the design stage and choose the correct structure of the composite material.

Currently, there are several basic methods for solving the aeroelasticity problem [1–6]. The most widely used method is based on the piston theory [1] (piston theory), which makes it possible to find an approximate solution to the aerodynamic problem, taking into account the deformation of the surface flowed around by the flow. After that, the solution of the problems of the theory of shells is corrected due to additional aeroelastic forces. More difficult to implement are the methods of conjugate solution of the problem of aerodynamics and the theory of elasticity [4]. As a rule, they are also solved with certain assumptions. The direct numerical solution of the adjoint problem of aeroelasticity is quite difficult to implement, since it requires the restructuring of computational grids for aerodynamic problems.

The purpose of this work was to apply the asymptotic theory to derive the equations of the theory of composite plates subjected to aeroelastic forces. A feature of this approach is that the general and three-dimensional equations of the theory of elasticity are chosen as the initial ones in this method, and the aerodynamic forces are calculated on the basis of the

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piston theory, but also in relation to the general three-dimensional theory of elasticity. The method of asymptotic expansions in terms of a small geometric parameter has shown itself well in other problems [7–13]. In the present work, this method is applied for the first time to problems of aeroelasticity.

2 Statement of the 3-dimensional problem of aeroelastic deformation

Consider a multilayer plate of constant thickness, introduce a small parameter $\kappa = h / L << 1$, as the ratio of the total thickness of the plate h to the characteristic size of the entire plate L (for example, to its maximum length). We also introduce global x_k and local ξ coordinates: $x_k = \tilde{x}_k / L,\ \xi = x_3 / \kappa,\ k = 1,2,3,$ where $\tilde{x}_k$ are Cartesian coordinates oriented in such a way that the axis $O\tilde{x}_3$ is directed along the normal to the outer and inner planes of the plate, and the axes $O\tilde{x}_1$ $O\tilde{x}_2$ belong to the middle surface of the plate. We believe that there are 2 scales of change in displacements u_k : one in directions of $O\tilde{x}_1$ $O\tilde{x}_2$, and the second in $O\tilde{x}_3$ direction. The coordinates x_3 and ξ, as usual, are treated as independent variables in the asymptotic averaging method. The coordinate ξ along the plate thickness varies in the range $-0.5 < \xi_3 < 0.5$.

Consider for a plate a 3-dimensional problem of the linear theory of aeroelasticity in the framework of small deformations and the linear piston theory [14]

$$\nabla_j \sigma_{ij} = \rho \ddot{u}_i$$

$$\varepsilon_{\mathrm{ij}} = \frac{1}{2}\left(\nabla_j u_i + \nabla_i u_j\right) \qquad (1)$$

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

$$\Sigma_{3\pm} : \sigma_{i3} = -\tilde{p}_{\pm}\delta_{i3}, \qquad \Sigma_T : u_i = u_{ei}, \qquad \Sigma_S : [\sigma_{i3}] = 0, \qquad [u_3] = 0,$$

consisting of the dynamic equations of the theory of linear elasticity, the Cauchy relations, the generalized Hooke's law, the boundary conditions on the outer surfaces of the shell plate - on the outer and inner surfaces $\Sigma_{3\pm}$ (their equation has the form $\tilde{x}_3 = \pm h / 2$) and on the end surface Σ_T, as well as the boundary conditions on the contact surface Σ_S of the layers of the plate, $[u_i]$ - jump of functions. In system (1), the following are indicated: σ_{ij} - stress tensor components, $\varepsilon_{\mathrm{ij}}$ - small strain tensor components, u_i - displacement vector components, C_{ijkl} - elastic modulus tensor components, ∇_i - covariant differentiation operator [15], - displacement vector components specified on Σ_T .

The outer surfaces $\Sigma_{3\pm}$ of the plate interact with a high-speed aerodynamic gas flow flowing around it with given constant velocity vectors $V_{\pm i}$. Then the pressure on these surfaces in the framework of the linearized piston theory is given in the form

$$\tilde{p}_{\pm} = \tilde{p}_{0\pm}(1 \pm q_A(\dot{u}_3 + V_{\pm I}\nabla_I u_3)). \tag{2}$$

Here, $\tilde{p}_{0\pm}$ is the pressure of the undisturbed aerodynamic flow on both parts of the plate surface, $q_A = \gamma_A / a_0$ is the aerodynamic coefficient of the plate, for which the expression is accepted, according to the linearized piston theory, where γ_A is the Poisson's ratio of the perfect gas model, a_0 is the speed of sound of the undisturbed gas flow.

3 Asymptotic expansions of the aeroelasticity problem

Problem (1) contains the local coordinate ξ , as well as a small parameter κ in the boundary conditions (this is the pressure coefficient), so we will seek its solution in the form of asymptotic expansions in the parameter in the form of functions depending on the global and local coordinates:

$$u_k = u_k^{(0)}(x_I) + \kappa u_k^{(1)}(x_I,\xi) + \kappa^2 u_k^{(2)}(x_I,\xi) + \kappa^3 u_k^{(3)}(x_I,\xi) + ... \tag{1}$$

Here and below, the indices denoted by capital letters I, J, K, L take the values 1,2, and the indices i, j, k, l - the values 1,2,3.

We substitute expansions (3) into the Cauchy relations in system (2), while using the rules of differentiation of local coordinate functions [10] $\partial / \partial\tilde{x}_j \rightarrow \partial / \partial x_j + (1/\kappa)\delta_{j3}\partial / \partial\xi$, then we obtain asymptotic expansions for deformations

$$\varepsilon_{ij} = \varepsilon_{ij}^{(0)} + \kappa\varepsilon_{ij}^{(1)} + \kappa^2\varepsilon_{ij}^{(2)} + ... \tag{2}$$

here the derivatives with respect to the local coordinate $u_{i/3}^{(1)} = \partial u_i^{(1)} / \partial\xi$ and with respect to the global coordinates $u_{i,j}^{(1)} = \partial u_i^{(1)} / \partial x_j$ are denoted.

Substituting expression (4) into Hooke's law in system (2), we obtain the asymptotic expansion for stresses

$$\sigma_{ij} = \sigma_{ij}^{(0)} + \kappa\sigma_{ij}^{(1)} + \kappa^2\sigma_{ij}^{(2)} + ... \tag{3}$$

For the pressure on the aerodynamic surfaces of the plate, we have the following decomposition

$$\begin{gathered}\tilde{p}_{\pm}=\kappa^{3}\tilde{p}_{\pm}^{(3)}+\kappa^{4}\tilde{p}_{\pm}^{(4)}\ldots,\\ \tilde{p}_{\pm}^{(3)}=p_{0\pm}(1\pm q_{A}(\dot{u}_{3}^{(0)}+V_{\pm I}u_{3,I}^{(0)}),\\ \tilde{p}_{\pm}^{(4)}=p_{0\pm}q_{A}(\dot{u}_{3}^{(1)}+V_{\pm I}u_{3,I}^{(1)})\end{gathered}\tag{6}$$

4 Formulation of local tasks

Substituting expansions (3)-(6) into the oscillation equations and the boundary conditions of system (1), we obtain

$$\frac{1}{\kappa}\sigma_{i3/3}^{(0)}+(\sigma_{iJ,J}^{(0)}+\sigma_{i3/3}^{(1)}-\rho\ddot{u}_{i}^{(0)})+\kappa(\sigma_{iJ,J}^{(1)}+\sigma_{i3/3}^{(2)}-\rho\ddot{u}_{i}^{(1)})+\kappa^{2}(\sigma_{iJ,J}^{(2)}+\sigma_{i3/3}^{(3)}-\rho\ddot{u}_{i}^{(2)})+\ldots=0,$$

$$\begin{aligned}\Sigma_{3\pm}:\sigma_{i3}^{(0)}+\kappa\sigma_{i3}^{(1)}+\kappa^{2}\sigma_{i3}^{(2)}+\kappa^{3}\sigma_{i3}^{(3)}+\ldots&=\\ =-\kappa^{3}p_{0\pm}(1\pm q_{A}(\dot{u}_{3}^{(0)}+V_{\pm I}u_{3,I}^{(0)}))\delta_{i3}&\mp\kappa^{4}p_{0\pm}q_{A}(\dot{u}_{3}^{(1)}+V_{\pm I}u_{3,I}^{(1)}))\delta_{i3}+\ldots,\end{aligned}\tag{7}$$

$$\Sigma_{T}:u_{i}=u_{i}^{(0)}+\kappa u_{i}^{(1)}+\kappa^{2}u_{i}^{(2)}+\kappa^{3}u_{i}^{(3)}+\ldots=u_{ei}$$

Equating the terms at κ^{-1} in the equilibrium equations to zero, and for the remaining powers of κ to some quantities $h_i^{(0)},h_i^{(1)},h_i^{(2)}$ independent of ξ_l, we obtain a recurrent sequence of local problems. The problem for the zero approximation has the form

$$\begin{aligned}&\sigma_{i3/3}^{(0)}=0,\\ &\sigma_{i3}^{(0)}=C_{i3KL}\varepsilon_{KL}^{(0)}+\tilde{C}_{i3k3}\varepsilon_{k3}^{(0)},\qquad \sigma_{IJ}^{(0)}=C_{IJKL}\varepsilon_{KL}^{(0)}+\tilde{C}_{IJk3}\varepsilon_{k3}^{(0)},\\ &\varepsilon_{IJ}^{(0)}=\frac{1}{2}(u_{I,J}^{(0)}+u_{J,I}^{(0)}),\qquad \varepsilon_{I3}^{(0)}=\frac{1}{2}(u_{3,I}^{(0)}+u_{I/3}^{(1)}),\qquad \varepsilon_{33}^{(0)}=u_{3/3}^{(1)},\\ &\Sigma_{3\pm}:\sigma_{i3}^{(0)}=0;\qquad \Sigma_{S}:[\sigma_{i3}^{(0)}]=0,\quad [u_{i}^{(1)}]=0,\quad <u_{i}^{(1)}>=0;\end{aligned}\tag{8}$$

The problem for the first approximation has the following form:

$$\begin{aligned}&\sigma_{i3/3}^{(1)}+\sigma_{iJ,J}^{(0)}+\rho\omega^{2}u_{i}^{(0)}=h_{i}^{(0)},\\ &\sigma_{i3}^{(1)}=C_{i3KL}\varepsilon_{KL}^{(1)}+\tilde{C}_{i3k3}\varepsilon_{k3}^{(1)},\qquad \sigma_{IJ}^{(1)}=C_{IJKL}\varepsilon_{KL}^{(1)}+\tilde{C}_{IJk3}\varepsilon_{k3}^{(1)},\\ &\varepsilon_{IJ}^{(1)}=\frac{1}{2}(u_{I,J}^{(1)}+u_{J,I}^{(1)}),\qquad \varepsilon_{I3}^{(1)}=\frac{1}{2}(u_{3,I}^{(1)}+u_{I/3}^{(2)}),\qquad \varepsilon_{33}^{(1)}=u_{3/3}^{(2)},\\ &\Sigma_{3\pm}:\sigma_{i3}^{(1)}=0;\qquad \Sigma_{S}:[\sigma_{i3}^{(1)}]=0,\quad [u_{i}^{(2)}]=0,\quad <u_{i}^{(2)}>=0;\end{aligned}\tag{9}$$

Problem for the second approximation

$$\begin{aligned}
&\sigma_{i3/3}^{(2)}+\sigma_{iJ,J}^{(1)}+\rho\omega^2u_i^{(1)}=h_i^{(1)},\\
&\sigma_{i3}^{(2)}=C_{i3KL}\varepsilon_{KL}^{(2)}+C_{i3k3}\varepsilon_{k3}^{(2)},\\
&\varepsilon_{IJ}^{(2)}=\frac{1}{2}(u_{I,J}^{(2)}+u_{J,I}^{(2)}),\qquad \varepsilon_{I3}^{(2)}=\frac{1}{2}(u_{3,I}^{(2)}+u_{I/3}^{(3)}),\qquad \varepsilon_{33}^{(2)}=u_{3/3}^{(3)},\\
&\Sigma_{3\pm}:\sigma_{i3}^{(2)}=0;\qquad \Sigma_S:[\sigma_{i3}^{(2)}]=0,\quad [u_i^{(3)}]=0,\qquad <u_i^{(3)}>=0;
\end{aligned}\tag{10}$$

Problem for the third approximation

$$\begin{aligned}
&\sigma_{i3/3}^{(3)}+\sigma_{iJ,J}^{(2)}+\rho\omega^2u_i^{(2)}=h_i^{(2)},\\
&\sigma_{i3}^{(3)}=C_{i3KL}\varepsilon_{KL}^{(3)}+C_{i3k3}\varepsilon_{k3}^{(3)},\\
&\varepsilon_{IJ}^{(3)}=\frac{1}{2}(u_{I,J}^{(3)}+u_{J,I}^{(2)}),\qquad \varepsilon_{I3}^{(3)}=\frac{1}{2}(u_{3,I}^{(3)}+u_{I/3}^{(4)}),\qquad \varepsilon_{33}^{(3)}=u_{3/3}^{(4)},\\
&\Sigma_{3\pm}:\sigma_{i3}^{(3)}=-p_{0\pm}(1\pm q_A(\dot u_3^{(0)}+V_{\pm I}u_{3,I}^{(0)}))\delta_{i3};\\
&\Sigma_S:[\sigma_{i3}^{(3)}]=0,\quad [u_i^{(4)}]=0,\\
&\quad <u_i^{(4)}>=0;
\end{aligned}\tag{11}$$

etc. Here and below, the operations of averaging over the plate thickness are denoted

$$<u_i^{(n)}>=\int_{-0.5}^{0.5}u_i^{(n)}d\xi\,,\ <f>_\xi=\int_{-0.5}^{\xi}fd\tilde\xi-<\int_{-0.5}^{\xi}fd\tilde\xi>\,,\ \{f\}_\xi=\int_{-0.5}^{\xi}(f-\langle f\rangle)d\tilde\xi$$

and also marked: $\tilde C_{ijK3}=2C_{ijK3}$, $\tilde C_{ij33}=C_{ij33}$.

The oscillation equations (7) after the introduction of the functions $h_i^{(0)},h_i^{(1)},h_i^{(2)}$, take the form

$$h_i^{(0)}+\kappa h_i^{(1)}+\kappa^2h_i^{(2)}+\ldots=0\,.\tag{12}$$

The solution of the local problem of zero approximation (8) are the functions $u_j^{(1)},\varepsilon_{kl}^{(0)},\sigma_{ij}^{(0)}$, they depend on the local coordinates ξ_l and the input data of this problem - displacements $u_j^{(0)}(x_J)$. The solution to problem (9) is the functions $u_j^{(2)},\varepsilon_{kl}^{(1)},\sigma_{ij}^{(1)}$, and $u_j^{(1)},\sigma_{ij}^{(0)}$ are the input data in this problem. In problem (10), the functions $u_j^{(3)},\varepsilon_{kl}^{(2)},\sigma_{ij}^{(2)}$ are unknown, and $u_j^{(2)},\varepsilon_{kl}^{(1)},\sigma_{ij}^{(1)}$ are the input data, and so on.

5 Solving the zero approximation problem

Due to the fact that problems (8)-(10) are one-dimensional with respect to the local variable , their solution can be found analytically. The solution of the equilibrium equations with boundary conditions in the local problem (8) has the form

$$\sigma_{i3}^{(0)} = 0 \tag{13}$$

Substituting into this equation the expression for $\sigma_{i3}^{(0)}$ from system (8), and taking into account the Cauchy equation of the same system for $\varepsilon_{k3}^{(0)}$, we obtain a differential equation with respect to $u_i^{(1)}$. The solution to this equation has the form

$$u_I^{(1)} = -\xi u_{3,I}^{(0)} + 2\varepsilon_{KL}^{(0)} U_{IKL}, \quad u_3^{(1)} = \varepsilon_{KL}^{(0)} U_{3KL}, \tag{14}$$

$$U_{iKL}(\xi) = -< Z_{iKL} >_{\xi} \quad Z_{iKL} = \tilde{C}_{i3j3}^{-1} C_{j3KL} .$$

Stresses $\sigma_{IJ}^{(0)}$, unlike $\sigma_{i3}^{(0)}$, are non-zero

$$\sigma_{IJ}^{(0)} = C_{IJKL}^{(0)} \varepsilon_{KL}^{(0)}, \tag{15}$$

$$C_{IJKL}^{(0)} = C_{IJKL} - C_{IJk3} \tilde{C}_{k3i3}^{-1} C_{i3KL} .$$

6 Solution of the problem of the first, second and third approximations

The solution of the equations of steady oscillations (9) - (11) together with the boundary conditions on Σ_S and $\xi = -0.5$ has the form

$$\sigma_{i3}^{(1)} = -\left\{\sigma_{iJ,J}^{(0)}\right\}_{\xi} + \ddot{u}_i^{(0)} \left\{\rho\right\}_{\xi},$$

$$\sigma_{i3}^{(2)} = -\left\{\sigma_{iJ,J}^{(1)}\right\}_{\xi} + \left\{\rho \ddot{u}_i^{(1)}\right\}_{\xi}, \tag{16}$$

$$\sigma_{i3}^{(3)} = -(\Delta p + p_A \dot{u}_3^{(0)} + p_I u_{3,I}^{(0)})\delta_{i3} - \left\{\sigma_{iJ,J}^{(2)}\right\}_{\xi} + \left\{\rho \ddot{u}_i^{(2)}\right\}_{\xi}$$

If we substitute expressions (15) into the first formula (16), we get

$$\sigma_{I3}^{(1)} = -\varepsilon_{KL,J}^{(0)} \left\{C_{IJKL}^{(0)}\right\}_{\xi} + \ddot{u}_I^{(0)} \left\{\rho\right\}_{\xi}, \quad \sigma_{33}^{(1)} = \ddot{u}_3^{(0)} \left\{\rho\right\}_{\xi} . \tag{17}$$

We express the deformations $\varepsilon_{k3}^{(1)}$ from the 2nd group of system (9), then, taking into account formulas (16), we obtain

$$\varepsilon_{k3}^{(1)} = -Z_{kKL}\varepsilon_{KL}^{(1)} - \varepsilon_{KL,J}^{(0)}\tilde{C}_{k3I3}^{-1}\left\{C_{IJKL}^{(0)}\right\}_{\xi} + \tilde{C}_{k3i3}^{-1}\ddot{u}_i^{(0)}\left\{\rho\right\}_{\xi}. \tag{18}$$

If we now substitute (18) into the 3rd group of relations (9), then we find the remaining stresses of the 1st approximation

$$\sigma_{IJ}^{(1)} = C_{IJKL}^{(0)}\varepsilon_{KL}^{(1)} + N_{IJKLM}^{(0)}\varepsilon_{KL,M}^{(0)} + G_{IJi}\ddot{u}_i^{(0)}, \tag{19}$$

$$N_{IJKLM}^{(0)} = -\tilde{C}_{IJk3}\tilde{C}_{k3P3}^{-1}\left\{C_{PMKL}^{(0)}\right\}_{\xi}, \quad G_{IJi} = \tilde{C}_{IJk3}\tilde{C}_{k3i3}^{-1}\left\{\rho\right\}_{\xi}.$$

Deformations $\varepsilon_{KL}^{(1)}$, taking into account formulas (9), (14), can be represented as

$$\varepsilon_{KL}^{(1)} = \xi\eta_{KL} + \Phi_{KLMNS}\varepsilon_{MN,S}^{(0)}, \tag{20}$$

$$\eta_{KL} = -u_{3,KL}^{(0)}, \quad \Phi_{KLMNS}(\xi) = U_{LMN}\delta_{KS} + U_{KMN}\delta_{LS}. \tag{21}$$

Taking into account formulas (20), expressions (19) take the form

$$\sigma_{IJ}^{(1)} = \xi C_{IJKL}^{(0)}\eta_{KL} + \tilde{N}_{IJKLM}^{(0)}\varepsilon_{KL,M}^{(0)} + G_{IJi}\ddot{u}_i^{(0)}, \tag{22}$$

$$\tilde{N}_{IJKLM}^{(0)} = N_{IJKLM}^{(0)} + C_{IJPQ}^{(0)}\Phi_{PQKLM}.$$

7 Averaged equations of steady oscillations of multilayer plates

Averaging the asymptotic expansion of the equations of motion (7) of the equilibrium plate, taking into account the boundary conditions, we obtain

$$\begin{aligned} &< \sigma_{iJ,J}^{(0)} > + < \rho > \ddot{u}_i^{(0)} + \kappa(< \sigma_{iJ,J}^{(1)} > + < \rho\ddot{u}_i^{(1)} >) + \\ &\quad + \kappa^2(< \sigma_{iJ,J}^{(2)} > - < \rho\ddot{u}_i^{(2)} > -(\Delta p + p_A\dot{u}_3^{(0)} + p_I u_{3,I}^{(0)})\delta_{i3}) + \ldots = 0 \end{aligned} \tag{23}$$

where are indicated

$$\Delta p = p_{0+} - p_{0-}, \quad p_A = q_A(p_{0+} + p_{0-}) \quad p_I = q_A(p_{0+}V_{+I} + p_{0-}V_{-I}).$$

We multiply the first equation of system (7) by $\xi\kappa$ and integrate them over the thickness, then we obtain the following auxiliary equation

$$\kappa(<\xi\sigma_{IJ}^{(0)}>_{,J}-<\rho\xi\ddot{u}_I^{(0)}>-<\sigma_{I3}^{(1)}>)+$$
$$+\kappa^2(<\xi\sigma_{IJ}^{(1)}>_{,J}-<\rho\ddot{u}_I^{(1)}\xi>-<\sigma_{I3}^{(2)}>)+...=0, \qquad (24)$$

Let us introduce the notation for forces T_{IJ}, moments M_{IJ} and shear forces Q_I in the plate

$$T_{IJ}=<\sigma_{IJ}^{(0)}>+\kappa<\sigma_{IJ}^{(1)}>+...,$$

$$Q_I=\kappa<\sigma_{I3}^{(1)}>+\kappa^2<\sigma_{I3}^{(2)}>+\left(\Delta p+p_A\dot{u}_3^{(0)}+p_Iu_{3,I}^{(0)}\right)\delta_{i3}+... \qquad (25)$$

$$M_{IJ}=\kappa<\xi\sigma_{IJ}^{(0)}>+\kappa^2<\xi\sigma_{IJ}^{(1)}>+... .$$

and also introduce the notation for the generalized displacements of the plate

$$\bar{\rho}U_i=<\rho>u_i^{(0)}+\kappa<\rho u_i^{(1)}>+\kappa^2<\rho u_i^{(2)}>+...$$

$$\bar{\rho}\Gamma_I=\kappa<\rho u_I^{(1)}\xi>+\kappa^2<\rho u_I^{(2)}\xi>+... \qquad (26)$$

where $\bar{\rho}=<\rho>$.

If only the main terms of the asymptotic expansions are retained in these expressions, then, taking into account (14), we obtain

$$U_i=u_i^{(0)}, \quad \bar{\rho}\Gamma_I=-Ru_{3,I}^{(0)}+\varepsilon_{KL}^{(0)}R_{IKL},$$

$$R_{IKL}=2\kappa\left\langle\xi U_{IKL}\right\rangle,\ R\ =\kappa<\rho\xi^2>. \qquad (27)$$

Taking into account the introduced notation (25) and (27), equations (23) and (24) can be written in the following form:

$$T_{IJ,J}=\bar{\rho}\ddot{u}_I^{(0)}, Q_{J,J}=\bar{\rho}\ddot{u}_3^{(0)}+\kappa^2\left(\Delta p+p_A\dot{u}_3^{(0)}+p_Iu_{3,I}^{(0)}\right), M_{IJ,J}=Q_I-R\ddot{u}_{3,I}^{(0)}+\ddot{u}_{K,L}^{(0)}R_{IKL}, \qquad (28)$$

We express Q_I from the third equation of system (26) and substitute into the second equation , as a result we get

$$M_{IJ,IJ}-\bar{\rho}\ddot{u}_3^{(0)}+R\ddot{u}_{3,II}^{(0)}-\ddot{u}_{K,IL}^{(0)}R_{IKL}-\left(\Delta\bar{p}+\bar{p}_A\dot{u}_3^{(0)}+\bar{p}_Iu_{3,I}^{(0)}\right)=0 \qquad (29)$$

- desired averaged equations of aeroelastic vibrations of a multilayer plate, here denoted: $\Delta\bar{p}=\kappa^2\Delta p$, $\bar{p}_A=\kappa^2p_A$, $\bar{p}_I=\kappa^2p_I$.

8 Average constitutive relations of plate theory

Substituting expressions (15) and (22) for stresses $\sigma_{IJ}^{(0)}$ and $\sigma_{IJ}^{(1)}$, into the integrals of formulas (25), we obtain

$$T_{IJ} = \bar{C}_{IJKL}\varepsilon_{KL}^{(0)} + B_{IJKL}\eta_{KL} + K_{IJKLM}\varepsilon_{KL,M}^{(0)} + \bar{G}_{IJi}\ddot{u}_i^{(0)}, \tag{30}$$

$$M_{IJ} = B_{IJKL}\varepsilon_{KL}^{(0)} + D_{IJKL}\eta_{KL} + \bar{K}_{IJKLM}\varepsilon_{KL,M}^{(0)} + \hat{G}_{IJi}\ddot{u}_i^{(0)}, \tag{31}$$

where are the tensors of the averaged elastic constants of the plate

$$\bar{C}_{IJKL} =< C_{IJKL}^{(0)} >=< C_{IJKL} > - < C_{IJk3}\tilde{C}_{k3i3}^{-1}\tilde{C}_{i3KL} >, \tag{32}$$

$$B_{IJKL} = \kappa < \xi C_{IJKL}^{(0)} >, \qquad K_{IJKLM} = \kappa < \tilde{N}_{IJKLM}^{(0)} >,$$

$$D_{IJKL} = \kappa^2 < \xi^2 C_{IJKL}^{(0)} >, \qquad \bar{K}_{IJKLM} = \kappa^2 < \xi\tilde{N}_{IJKLM}^{(0)} >,$$

$$\bar{G}_{IJi} = \kappa < G_{IJi} >, \quad \hat{G}_{IJi} = \kappa^2 < \xi G_{IJi} >,$$

9 Closed system of equations for aeroelastic deformations of a multilayer plate

The system of averaged constitutive relations (30)-(32) includes deformations $\varepsilon_{KL}^{(0)}$ of the middle surface , curvatures η_{KL} and gradients of deformations $\varepsilon_{KL,N}^{(0)}$, which depend on 3 functions $u_I^{(0)}$, $u_3^{(0)}$ of global variables x_I,

$$\varepsilon_{IJ}^{(0)} = \frac{1}{2}(u_{I,J}^{(0)} + u_{J,I}^{(0)}), \qquad \eta_{KL} = -u_{3,KL}^{(0)}. \tag{33}$$

Substituting further relations (30), (31) and (33) into (28), (29) we obtain a system of 3 equations for 3 unknown displacements:

$$\bar{C}_{IJKL}u_{K,LJ}^{(0)} - B_{IJKL}u_{3,KLJ}^{(0)} + K_{IJKLM}u_{K,LMJ}^{(0)} + \bar{G}_{IJi}\ddot{u}_{i,J}^{(0)} - \bar{\rho}\ddot{u}_I^{(0)} = 0,$$

$$\begin{aligned} &B_{IJKL}u_{K,LJI}^{(0)} - D_{IJKL}u_{3,KLJI}^{(0)} + \bar{K}_{IJKLM}u_{K,LMJI}^{(0)} - \bar{\rho}\ddot{u}_3^{(0)} + \\ &\quad + (R\delta_{IJ} + \hat{G}_{IJ3})\ddot{u}_{3,IJ}^{(0)} - \ddot{u}_{K,IL}^{(0)}(R_{IKL} - \hat{G}_{ILK}) - \left(\Delta\bar{p} + \bar{p}_A\dot{u}_3^{(0)} + \bar{p}_I u_{3,I}^{(0)}\right) = 0 \end{aligned} \tag{34}$$

This system has the fourth order of derivatives with respect to the coordinates with respect to deflection $u_3^{(0)}$, as in the classical theory of Kirchhoff-Love plates, and the third order of derivatives with respect to longitudinal displacements $u_I^{(0)}$, and also contains mixed derivatives $\ddot{u}_{i,IJ}^{(0)}$ of the 4th order. The aeroelastic terms contain derivatives at the lowest derivatives - the first order $\dot{u}_3^{(0)}$ and $u_{3,I}^{(0)}$.

10 Solution of the problem of flexural aeroelastic vibrations of a composite plate

Let us consider harmonic bending aeroelastic vibrations of a symmetric orthotropic multilayer plate made of composite material, when there are no longitudinal movements

$$u_I^{(0)} \equiv 0 \ , \ u_3^{(0)} = u_3^{(0)}(x_1) \tag{35}$$

Then the system of equations (34) is reduced to one equation of aeroelastic vibrations:

$$D_{1111}u_{3,1111}^{(0)} + \Delta\bar{p} + \bar{p}_A\dot{u}_3^{(0)} + \bar{p}_1 u_{3,1}^{(0)} - (R + G_{113})\ddot{u}_{3,11}^{(0)} + \bar{\rho}\ddot{u}^{(0)} = 0 \tag{36}$$

Let us consider the case when the pressure on the plate surface changes according to the harmonic law $\Delta\bar{p} = \Delta p' \cos\omega t$, $\bar{p}_A = 0$. Then the solution of Eq. (36) will be sought in the following form

$$u_3^{(0)} = u(x)\cos\omega t \tag{37}$$

From (36) we obtain the following equation for calculating the amplitude of transverse oscillations

$$D_{1111}u_{,1111} + p_1 u_{,1} + (R + G_{113})\omega^2 u_{,11} - \bar{\rho}\omega^2 u + \Delta\bar{p}' = 0 \tag{38}$$

The solution of the differential equation (38) with the hinged boundary conditions was implemented in a program in the python programming language. The calculations were carried out for a composite material based on glass fibers and an epoxy matrix with the following parameters: : $h = 2\cdot 10^{-4}$ m, $L = 1$ m, $\rho_m = 1000\frac{\kappa g}{м^3}$ - matrix density, $\rho_f = 2000\frac{\kappa g}{м^3}$ - glass fiber density, $\varphi_f = 0.6$- composite reinforcement coefficient, $E_m = 1GPa$ - matrix elasticity modulus, $E_f = 200,\ ГПа$ - fiber elasticity modulus.

The constants D_{1111}, R and $\bar{\rho}$ included in the differential equation (34) were calculated from the values of the elastic moduli C_{ijkl}, which, in turn, were calculated using the method from [16] for the values of the constants of the epoxy matrix and glass fibers using the SMCM software package implemented in the c++ programming language.

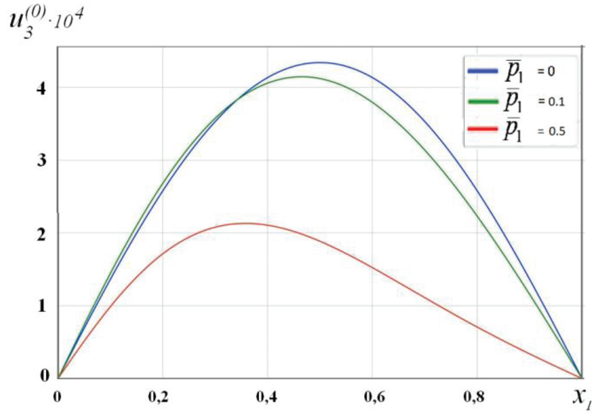

Fig. 1. Dependence of composite plate deflection on the aeroelastic pressure coefficient at zero frequency.

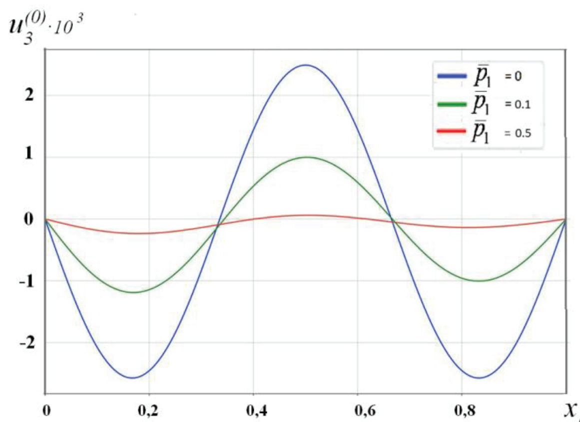

Fig. 2. Dependence of composite plate deflection on the aeroelastic pressure coefficient at zero frequency.

The dependence of the transverse displacements of the composite material on the aeroelastic pressure coefficient $\bar{p}_1$ was studied. Calculations were carried out in the absence of aeroelastic oscillations $\omega = 0$, when the effect of the gas flow on the plate is caused only by aeroelastic quasi-stationary deformation, determined by the coefficient $\bar{p}_1$ (Fig. 1). As the coefficient $\bar{p}_1$ increases, the maximum plate deflection decreases, which is due to the appearance of additional plate rigidity caused by aeroelastic interaction with the gas flow.

Figure 2 shows the results of the calculation of plate aeroelastic vibrations under $\omega = 0.1s$ at for various values of the coefficient $\bar{p}_1$. In this case, with an increase in the coefficient $\bar{p}_1$, the amplitude of the plate oscillations also decreases due to the appearance of additional plate rigidity caused by taking into account the resistance of the gas flow.

All values of deflection $u(x)$, coordinates and parameters $\bar{p}_1$ and Δp - are dimensionless.

11 Conclusions

A theory for modeling aeroelastic deformations of a composite thin plate is proposed based on the asymptotic theory using the piston theory as boundary conditions.

The basic equations of the asymptotic theory are obtained - a system of averaged equations for aeroelastic vibrations of a plate, and a sequence of local problems. For local problems, a solution is obtained in an explicit analytical form.

As an example, calculations of vibrational aeroelastic oscillations of a thin plate made of composite material are carried out and the dependence of oscillations on the pressure of an external gas flow is studied. It is shown that with an increase in the coefficient of aeroelastic pressure of the gas flow, the additional rigidity of the plate increases, as a result of which the amplitude of the plate deflection decreases.

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