Modeling the stress-strain state of variable-thickness composite shells and plates

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Abstract. A model for calculating the stress-strain state of variable-thickness composite shells has been developed, based on assumptions such us the classical theory of Timoshenko-Mindlin shells. In the proposed model, the plate thickness is given by a function of curvilinear coordinates and is directly considered in the derivation of the equilibrium equations of the plate. The general equations of the theory of variable-thickness composite plates are derived. The article analyses the solution of the problem of plates bending under uniform pressure considering the variable thickness. For the numerical solution, the finite difference method (FDM) has been applied to the system of differential equations with matrix coefficients. For the resultant algebraic system, the FDM uses the tridiagonal matrix algorithm in computing the solution. The calculation results are compared with a plate of constant thickness. It is shown that the effect of thickness variability is quite significant. **Key words:** variable-thickness plates, theory of Timoshenko-Mindlin shells, tridiagonal matrix algorithm.

1 Introduction

Currently, there is a great interest in the modeling of variable-thickness composite plates. Examples of such plates are aircraft wings, composite springs, ship hulls made from composites, and other structural elements. Variable thickness can be used to increase strength in specific areas of the wall or bottom [1], without unnecessary mass cost to the rest of the structure. In addition, variable thickness can occur in the presence of object defects [2, 3], for example, when a structure is made during curing [4, 5]. Accounting for variable thickness can improve the quality of the product and reduce the cost of manufacturing structures. Analysis of the behavior of stiffened composite panels with variable thickness skin under to uniaxial compression can be used to predict the bending load [6].

There are various approaches to the modeling of variable-thickness plates. The work [7] considers a flat plate, symmetric about the midplane with a variable thickness parameter, which is modeled by a set of control points through NURBS basic functions. Strain-displacement relations in sense of von-Karman theory are employed for large deformation. In the articles of Firsanov V.V. [8-11], the calculation of the stress-strain state for plates and shells with asymmetric variable thickness is considered, taking into account the influence of the boundary layer effect in boundary value problems. In [12], the stability of a multilayer

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round plates of variable thickness from nonlinear elastic material is studied. In order to improve the methods of calculating stability, the author considers plates with functional graduation subjected to radical compression. The study was carried out on the basis of the theory of plates of first order shear deformations and the field of nonlinear von Karman displacements.

In [13] isotropic plates and composite laminated plates are modeled using variable thickness, shear deformable, finite plate elements under different types of compressive loads. Improvements in both uniaxial and biaxial buckling loads of over 160% are shown possible in composite plates compared to uniform quasiisotropic plates.

In papers [14-15], the problems of thermoelasticity of variable thickness rotating disks are considered. In [14], the disk thickness varies according to a power law, and a semi-analytical solution for the displacement was obtained. In [15], based on the classical theory of plates and first order shear deformations theory, a linear and nonlinear analysis is carried out with different thickness-to-radius ratios to account for the influence of shear deformation and nonlinearity.

In [16], the stress-strain state of nonlinear elastic orthotropic thin shells with stiffened holes and shells of discretely variable thickness was studied. The reduction surface is not necessary the median surface. The constitutive equations are obtained basing on the Lomakin theory of plasticity of anisotropic media.

In this paper, a method for deriving a closed system of equations in the theory of composite shells is developed, which generalizes the classical Timoshenko-Mindlin shell theory. A description of this theory can be found in [17]. From the equations of the general three-dimensional theory of elasticity in curvilinear coordinates, the equilibrium equations, the Cauchy relations and the elasticity for shells with variable thickness are derived. A complete closed system of equations for flat variable thickness plates is formulated. As an example, the problem of bending a plate loaded with uniform pressure is considered.

2 Mathematical formulation of the problem of elasticity variable-thickness plates

Let us consider the quasi-static problem of the three-dimensional linear elasticity theory of elasticity in the general formulation. We write the equations included in this system in orthogonal coordinatesX^i, two of which X^α, $\alpha = 1{,}2$ two of which coincide with the lines of the principal curvatures of some base surface Σ_0 of the structure, the coordinate X^3 is oriented along the normal to this surface Σ_0.

The equilibrium equations in orthogonal coordinates X^i have the form [18]

$$\begin{gathered}\left(H_\beta H_\gamma \sigma_{\alpha\alpha}\right)_{,\alpha} + \left(H_\alpha H_\gamma \sigma_{\alpha\beta}\right)_{,\beta} + \left(H_\alpha H_\beta \sigma_{\alpha\gamma}\right)_{,\gamma} + \sigma_{\alpha\beta} H_\gamma H_{\alpha,\beta} + \sigma_{\alpha\gamma} H_\beta H_{\alpha,\gamma} - \\ -\sigma_{\beta\beta} H_\gamma H_{\beta,\alpha} - \sigma_{\gamma\gamma} H_\beta H_{\gamma,\alpha} + H_1 H_2 H_3 \mathring{\rho} f_\alpha = 0, \quad \alpha,\beta,\gamma = 1{,}2{,}3, \alpha \neq \beta \neq \gamma \neq \alpha, \end{gathered} \tag{1}$$

where $\sigma_{\alpha\beta}$ - physical components of the stress tensor in the basis, associated with coordinates X^i, H_γ- Lame parameters for coordinates X^i, $H_{\alpha,\beta} = \frac{\partial H_\alpha}{\partial X^\beta}$ - partial derivatives, $\mathring{\rho}$- density, f_α- mass force density.

The Cauchy relations in the curvilinear coordinate system have the form [18]:

$$\begin{gathered}\varepsilon_{\alpha\alpha} = O_\alpha u_{\alpha,\alpha} + O_\alpha O_\beta H_{\alpha,\beta} u_\beta + O_\alpha O_\gamma H_{\alpha,\gamma} u_\gamma, \\ 2\varepsilon_{\alpha\beta} = H_\alpha O_\beta (O_\alpha u_\alpha)_{,\beta} + H_\beta O_\alpha (O_\beta u_\beta)_{,\alpha}; \quad \alpha,\beta,\gamma = 1{,}2{,}3, \alpha \neq \beta \neq \gamma \neq \alpha, \end{gathered} \tag{2}$$

where $\varepsilon_{\alpha\beta}$- small-strain tensor components, u_α - displacement vector components, $O_\alpha = 1/H_\alpha$.

The stress-strain relations are written as follows:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \tag{3}$$

where C_{ijkl}- components of the elastic modulus tensor.

We assume that the shell is thin and make the following assumptions [18],

$$H_3 = 1;\ H_\alpha = A_\alpha;\ \ H_{\alpha,3} = \bar{k}_\alpha A_\alpha; \quad X^3\bar{k}_\alpha << 1,\ \alpha = 1,2, \tag{4}$$

where A_α coefficients of the first quadratic form of the base surface Σ_0, a $\bar{k}_\alpha$- her principal curvatures.

As in the classical theory of Timoshenko-Mindlin shells, the displacements u_α in the shell will be considered as linear functions of the coordinate X^3, u_3 - independent of the coordinate X^3

$$u_\alpha = U_\alpha + X^3\gamma_\alpha, \quad u_3 = W, \quad \alpha = 1,2; \tag{5}$$

where U_α - displacements, γ_α normal rotation angles, W – shell deflection.

The normal stress σ_{33} in the shell is neglected:

$$\sigma_{33} = 0 \tag{6}$$

For a variable thickness shell, the coordinate X^3 takes the values $-a^-(X^1, X^2) < X^3 < a^+(X^1, X^2)$, where $X^3 = a^\pm(X^1, X^2)$ – equations of the outer and inner surface of the shell. The function $h = a^+(X^1, X^2) + a^-(X^1, X^2)$ is the thickness of the shell and in this work is a variable depending on the coordinates X^1, X^2.

Substituting (5) into (2), we obtain expressions for deformations in the shell, coinciding with the equations of the classical Timoshenko theory

$$\varepsilon_{\alpha\beta} = e_{\alpha\beta} + X^3\varkappa_{\alpha\beta}, \quad \varepsilon_{33} = 0, \quad \varepsilon_{\alpha3} = e_{\alpha3}, \alpha = 1,2; \tag{7}$$

where e_{11}, e_{22}, e_{12}- strains, $\varkappa_{11}, \varkappa_{22}, \varkappa_{12}$- curvatures of the middle surface and $e_{\alpha3}$ - shear strains. All of these functions are expressed in terms of U_α,γ_α and W [18].

Assuming the shell is orthotropic, the component representation of the stress-strain relations (3) given (7) has the form:

$$\begin{gathered}\sigma_{\alpha\alpha} = C_{\alpha\alpha}e_{\alpha\alpha} + C_{\alpha\beta}e_{\beta\beta} + X^3\left(C_{\alpha\alpha}\varkappa_{\alpha\alpha} + C_{\alpha\beta}\varkappa_{\beta\beta}\right), \quad \alpha = 1,2,\\ \sigma_{12} = 2C_{66}e_{12} + 2X^3C_{66}\varkappa_{12}, \quad \sigma_{23} = 2C_{44}e_{23}, \quad \sigma_{13} = 2C_{55}e_{13}.\end{gathered} \tag{8}$$

where $C_{\alpha\beta}$ are the components of the elasticity modulus matrix, expressed in terms of the components C_{ijkl} [18], 6x6 in size.

The resultant forces $T_{\alpha\beta}$, moments$M_{\alpha\beta}$ and shear forces Q_α are obtained by integration of the stresses through the variable thickness of composite shells:

$$T_{\alpha\beta} = \int_{-a^-}^{a^+} \sigma_{\alpha\beta}\, dX^3,\ M_{\alpha\beta} = \int_{-a^-}^{a^+} \sigma_{\alpha\beta}\, X^3 dX^3,\ Q_\alpha = \int_{-a^-}^{a^+} \sigma_{\alpha3}\, dX^3, \quad \alpha = 1,2. \tag{9}$$

Mass forces $F_{e\alpha}$ and mass moments$M_{e\alpha}$ are introduced by the formulas:

$$F_{e\alpha} = \int_{-a^-}^{a^+} \overset{\circ}{\rho} f_\alpha dX^3 \, , \; M_{e\alpha} = \int_{-a^-}^{a^+} \overset{\circ}{\rho} f_\alpha X^3 dX^3, \tag{10}$$

The boundary conditions on the shell surfaces $X^3 = \pm a^\pm$ are assumed to be force only (given pressures$p_e^\pm$):

$$X^3 = \pm a^\pm \colon \; \sigma_{33}^\pm = -p_e^\pm, \; \sigma_{\alpha 3}^\pm = 0, \; \alpha\text{=}1{,}2, \tag{11}$$

Displacements can be specified on one part of the end surface of the shell, and the components of the stress vector can be specified on the remaining part.

3 Derivation of equilibrium equations considering the variable-thickness of the shell

To obtain the equilibrium equations, we use the formula for the differentiation of the integral with respect to the parameter in the case of variable integration limits

$$F(X^1, X^2)_{,\alpha} = \int_{-a^-}^{a^+} \big(f(X^1, X^2, X^3)\big)_{,\alpha} dX^3 + f(X^1, X^2, a^+) a_{,\alpha}^+ + f(X^1, X^2, -a^-) a_{,\alpha}^- \tag{12}$$

We integrate equation (1) over X^3, setting $\alpha = 3, \beta = 1, \gamma = 2$, and using formulas (4), with boundary conditions (11) and $H_{31} = H_{32} = 0$

$$\int_{-a^-}^{a^+} (A_2 A_1 \sigma_{33})_{,3} dX^3 + \left(\int_{-a^-}^{a^+} A_2 \sigma_{13} dX^3 \right)_{,1} + \left(\int_{-a^-}^{a^+} A_1 \sigma_{23} dX^3 \right)_{,2} - \\ - \int_{-a^-}^{a^+} A_1 A_2 \bar{k}_1 \sigma_{11} dX^3 - \int_{-a^-}^{a^+} A_1 A_2 \bar{k}_2 \sigma_{22} dX^3 + \int_{-a^-}^{a^+} A_1 A_2 \overset{\circ}{\rho} f_\alpha dX^3 = 0. \tag{13}$$

Similarly, we transform the second equation in (1), setting $\alpha = 1, \beta = 2, \gamma = 3$

$$\left(\int_{-a^-}^{+a^+} A_2 \sigma_{11} dX^3 \right)_{,1} - A_2 \sigma_{11}^+ (a^+)_{,1} - A_2 \sigma_{11}^- (a^-)_{,1} + \left(\int_{-a^-}^{a^+} A_1 \sigma_{12} dX^3 \right)_{,2} - A_1 \sigma_{12}^+ (a^+)_{,2} - \\ - A_1 \sigma_{12} \;\; (a^-)_{,2} - \int_{-a^-}^{a^+} \sigma_{22} A_{2,1} dX^3 + \int_{-a^-}^{a^+} \sigma_{12} A_{1,2} dX^3 + \int_{-a^-}^{a^+} A_1 A_2 \bar{k}_1 \sigma_{13} dX^3 + \\ \int_{-a^-}^{a^+} A_1 A_2 \overset{\circ}{\rho} f_1 dX^3 = 0. \tag{14}$$

The third equilibrium equation is obtained for $\alpha = 2, \beta = 3, \gamma = 1$:

$$\left(\int_{-a^-}^{+a^+} A_1 \sigma_{22} dX^3 \right)_{,2} - A_1 \sigma_{22}^+ (a^+)_{,2} - A_1 \sigma_{22}^- (a^-)_{,2} + \left(\int_{-a^-}^{+a^+} A_2 \sigma_{21} dX^3 \right)_{,1} - A_2 \sigma_{21}^+ (a^+)_{,1} \\ - \\ - A_2 \sigma_{21}^- (a^-)_{,1} + + \int_{-a^-}^{a^+} A_1 A_2 \bar{k}_2 \sigma_{23} dX^3 + \int_{-a^-}^{a^+} \sigma_{21} A_{2,1} dX^3 - \int_{-a^-}^{a^+} \sigma_{11} A_{1,2} dX^3 + \\ \int_{-a^-}^{a^+} A_1 A_2 \overset{\circ}{\rho} f_2 dX^3 = 0. \tag{15}$$

We obtain the fourth equilibrium equation with moments by multiplying (1) by X^3 for $\alpha = 1, \beta = 2, \gamma = 3$ and then integrating it over X^3:

$$\left(\int_{-a^-}^{a^+} A_2\sigma_{11}X^3dX^3\right)_{,1} - A_2a^+\sigma_{11}^+(a^+)_{,1} + A_2a^-\sigma_{11}^-(a^-)_{,1} + \left(\int_{-a^-}^{a^+} A_1\sigma_{12}X^3dX^3\right)_{,2} - A_1a^+\sigma_{12}^+(a^+)_{,2} + A_1a^-\sigma_{12}^-(a^-)_{,2} - \int_{-a^-}^{a^+} \sigma_{22}A_{2,1}X^3dX^3 + \int_{-a^-}^{a^+} \sigma_{12}A_{1,2}X^3dX^3 + \int_{-a^-}^{a^+} A_1A_2\mathring{\rho}f_1X^3dX^3 = 0. \quad (16)$$

Similarly, we obtain the fifth equilibrium equation for $\alpha = 2, \beta = 3, \gamma = 1$:

$$\left(\int_{-a^-}^{a^+} A_1\sigma_{22}X^3dX^3\right)_{,2} - A_1\sigma_{22}^+a^+(a^+)_{,2} + A_1\sigma_{22}^-a^-(a^-)_{,2} + \left(\int_{-a^-}^{a^+} A_2\sigma_{21}X^3dX^3\right)_{,1} - A_2\sigma_{21}^+a^+(a^+)_{,1} + +A_2\sigma_{21}^-a^-(a^-)_{,1} - \int_{-a^-}^{a^+}(A_1A_2\sigma_{23})\,dX^3 + \int_{-a^-}^{a^+}\sigma_{21}A_{2,1}X^3dX^3 - \int_{-a^-}^{a^+}\sigma_{11}A_{1,2}X^3dX^3 + + \int_{-a^-}^{a^+} A_1A_2\mathring{\rho}f_2X^3dX^3 = 0. \quad (17)$$

Thus, we obtain five equilibrium equations considering the variable thickness of the shells.

4 Equilibrium equations for the shells of the variable-thickness

We write equations (13-17) for shells using integral relations for forces and moments (9-10):

$$-A_1A_2(p_e^+ - p_e^-) + (A_2Q_1)_{,1} + (A_1Q_2)_{,2} - A_1A_2\left(\bar{k}_1T_{11} + \bar{k}_2T_{22}\right) + A_1A_2F_{e3} = 0 \quad (18)$$

$$(A_2T_{11})_{,1} + (A_1T_{12})_{,2} - T_{22}A_{2,1} + T_{12}A_{1,2} - A_2\sigma_{11}^+(a^+)_{,1} - A_2\sigma_{11}^-(a^-)_{,1} \quad (19)$$

$$(A_1T_{22})_{,2} + (A_2T_{21})_{,1} + T_{21}A_{2,1} - T_{11}A_{1,2} + A_1A_2\bar{k}_2Q_2 - -A_1\sigma_{22}^+(a^+)_{,2} - A_1\sigma_{22}^-(a^-)_{,2} - A_2\sigma_{21}^+(a^+)_{,1} - A_2\sigma_{21}^-(a^-)_{,1} + A_1A_2F_{e2} = 0 \quad (20)$$

$$(A_2M_{11})_{,1} + (A_1M_{12})_{,2} - A_1A_2Q_1 - M_{22}A_{2,1} + M_{12}A_{1,2} - -A_2a^+\sigma_{11}^+(a^+)_{,1} + A_2a^-\sigma_{11}^-(a^-)_{,1} - A_1a^+\sigma_{12}^+(a^+)_{,2} + A_1a^-\sigma_{12}^-(a^-)_{,2} + A_1A_2M_{e1} = 0 \quad (21)$$

$$(A_1M_{22})_{,2} + (A_2M_{21})_{,1} - A_1A_2Q_2 + M_{21}A_{2,1} - M_{11}A_{1,2} - -A_1a^+\sigma_{22}^+(a^+)_{,2} + A_1a^-\sigma_{22}^-(a^-)_{,2} - A_2a^+\sigma_{21}^+(a^+)_{,1} + A_2a^-\sigma_{21}^-(a^-)_{,1}{}^{-} + A_1A_2M_{e2} = 0 \quad (22)$$

5 The stress-strain relations of variable-thickness composite shells

Substituting expressions (8) into (9), we find the relationship between forces, moments, shear forces and deformations $e_{\alpha\beta}$, $\varkappa_{\alpha\beta}$:

$$T_{11} = \bar{C}_{11}e_{11} + \bar{C}_{12}e_{22} + B_{11}\varkappa_{11} + B_{12}\varkappa_{22},$$

$$T_{22} = \bar{C}_{22}e_{22} + \bar{C}_{21}e_{11} + B_{22}\varkappa_{22} + B_{21}\varkappa_{11},$$

$$T_{12} = 2\bar{C}_{66}e_{12} + 2B_{66}\varkappa_{12}, T_{21} = 2\bar{C}_{66}e_{21} + 2B_{66}\varkappa_{21},$$

$$M_{11} = B_{11}e_{11} + B_{12}e_{22} + D_{11}\varkappa_{11} + D_{12}\varkappa_{22}, \tag{23}$$

$$M_{22} = B_{22}e_{22} + B_{21}e_{11} + D_{22}\varkappa_{22} + D_{21}\varkappa_{11},$$

$$M_{12} = 2B_{66}e_{12} + 2D_{66}\varkappa_{12}, M_{21} = 2B_{66}e_{21} + 2D_{66}\varkappa_{21},$$

$$Q_1 = 2\bar{\mathrm{C}}_{55}e_{13}, Q_2 = 2\bar{\mathrm{C}}_{44}e_{23}.$$

The reduced stiffnesses are defined by the formulas

$$\bar{C}_{\alpha\beta} = \int_{-a^-}^{a^+} C_{\alpha\beta}dX^3\,,\; B_{\alpha\beta} = \int_{-a^-}^{a^+} C_{\alpha\beta}X^3\,dX^3,\;\; D_{\alpha\beta} = \int_{-a^-}^{a^+} C_{\alpha\beta}(X^3)^2\,dX^3, \tag{24}$$

Due to the variability of the shell thickness, the equilibrium equations (19)-(22) include additional unknowns - $\sigma_{\alpha\beta}^{\pm}$- stresses on the outer and inner surfaces of the shell. Let us find additional stress-strain relations that connect these stresses with deformations and curvatures $e_{\alpha\beta}$, $\varkappa_{\alpha\beta}$ of the base surface of the shell.

Considering formulas (7) and (8), we calculate the values of strains $\varepsilon_{\alpha\beta}^{\pm}$ and stresses $\sigma_{\alpha\beta}^{\pm}$ on the outer surfaces of the shell:

$$\varepsilon_{11}^{\pm} = e_{11} \pm a^{\pm}\varkappa_{11},$$

$$\varepsilon_{22}^{\pm} = e_{22} \pm a^{\pm}\varkappa_{22}, \tag{25}$$

$$\varepsilon_{12}^{\pm} = e_{12} \pm a^{\pm}\varkappa_{12},$$

$$\sigma_{11}^{\pm} = C^{\pm}{}_{11}e_{11} + C^{\pm}{}_{12}e_{22} \pm a^{\pm}(C^{\pm}{}_{11}\varkappa_{11} + C^{\pm}{}_{12}\varkappa_{22}),$$

$$\sigma_{22}^{\pm} = C^{\pm}{}_{22}e_{22} + C^{\pm}{}_{21}e_{11} \pm a^{\pm}(C^{\pm}{}_{22}\varkappa_{22} + C^{\pm}{}_{21}\varkappa_{11}), \tag{26}$$

$$\sigma_{12}^{\pm} = 2C^{\pm}{}_{66}e_{12} \pm 2a^{\pm}C^{\pm}{}_{66}\varkappa_{12}.$$

6 Solution of the problem of bending a variable-thickness plate

Using the described technique, we solve the problem of bending a variable-thickness plate. In this case, the elasticity equations depend only on the coordinate $\boldsymbol{X^1}$ and the number of equilibrium equations becomes three:

$$(Q_1)_{,1} = \Delta p - F_{e3},\; \Delta p = p^+ - p^-,$$

$$(T_{11})_{,1} - \sigma_{11}^{+}(a^+)_{,1} - \sigma_{11}^{-}(a^-)_{,1} + F_{e1} = 0, \tag{27}$$

$$(M_{11})_{,1} = Q_1 - M_{e1} + a^+\sigma_{11}^{+}(a^+)_{,1} - a^-\sigma_{11}^{-}(a^-)_{,1}.$$

The relationships between strains and displacements are defined as

$$e_{11} = U_{1,1}, \quad \varkappa_{11} = \gamma_{1,1}, \quad 2e_{13} = \gamma_1 + W_{,1}, \tag{28}$$

Formulas (23) and (26) take the form

$$T_{11} = \bar{C}_{11}e_{11} + B_{11}\varkappa_{11}, \quad M_{11} = B_{11}e_{11} + D_{11}\varkappa_{11}, \quad Q_1 = 2\overline{C_{55}}e_{13},$$

$$\sigma_{11}^{\pm} = C^{\pm}{}_{11}(e_{11} \pm a^{\pm}\varkappa_{11}), \tag{29}$$

The plate is loaded with uniform pressure, the left edge is rigidly clamped, the right one is free:

$$X^1 = 0: \ U_1 = 0, \ \gamma_1 = 0, \ W = 0,$$
$$X^1 = l: \ T_{11} = 0, \ M_{11} = 0, \ Q_1 = 0,$$

$$M_{e1} = 0, \quad \Delta p = const, F_{e1} = F_{e3} = 0 \tag{30}$$

Let us express the equilibrium equations (27) in terms of displacements U_1, W, γ_1 and replace them with the expressions for the integral characteristics T_{11}, M_{11}, Q_1 (29) expressed in terms of displacements, taking into account (28). We obtain a system of three second-order differential equations.

$$\begin{cases} (W_{,11}\bar{C}_{55}) + [\gamma_{1,1}\bar{C}_{55} + W_{,1}\bar{C}_{55,1}] + \gamma_1\bar{C}_{55,1} = \Delta p, \\ (U_{1,11}\bar{C}_{11} + B_{11}\gamma_{1,11}) + [\gamma_{1,1}(S_1 + B_{11,1}) + U_{1,1}(S_2 + \bar{C}_{11,1})] = 0, \\ (D_{11}\gamma_{1,11} + B_{11}U_{1,11}) + [\gamma_{1,1}(S_3 + D_{11,1}) - W_{,1}\bar{C}_{55} + U_{1,1}(S_1 + B_{11,1})] - \gamma_1\bar{C}_{55} = 0. \end{cases} \tag{31}$$

where

$$\bar{C}_{\alpha\beta,1} = C_{\alpha\beta}{}^{+}a^{+} + C_{\alpha\beta}{}^{-}a^{-}; B_{\alpha\beta,1} = \left(C_{\alpha\beta}{}^{+}a^{+}(a^{+})_{,1} - C_{\alpha\beta}{}^{-}a^{-}(a^{-})_{,1}\right);$$
$$D_{\alpha\beta,1} = \left(C_{\alpha\beta}{}^{+}(a^{+})^2(a^{+})_{,1} + C_{\alpha\beta}{}^{-}(a^{-})^2(a^{-})_{,1}\right). \tag{32}$$

$$S_1 = -a^{+}C_{11}{}^{+}a^{+}{}_{,1} + a^{-}C_{11}{}^{-}a^{-}{}_{,1}; \ S_2 = -C_{11}{}^{+}a^{+}{}_{,1} - C_{11}{}^{-}a^{-}{}_{,1};$$
$$S_3 = -(a^{+})^2C_{11}{}^{+}a^{+}{}_{,1} - (a^{-})^2C_{11}{}^{-}a^{-}{}_{,1}.$$

We write system (32) as a second-order matrix differential equation with respect to the variable $\boldsymbol{X^1}$:

$$[K_1]\{y\}'' + [K_2]\{y\}' + [K_3]\{y\} = -\{K_4\}, \tag{33}$$

Where

$$y = \begin{pmatrix} \gamma \\ W \\ U \end{pmatrix}, [K_1] = \begin{pmatrix} 0 & \bar{C}_{55} & 0 \\ B_{11} & 0 & \bar{C}_{11} \\ D_{11} & 0 & B_{11} \end{pmatrix}, [K_2] = \begin{pmatrix} \bar{C}_{55} & \bar{C}_{55,1} & 0 \\ S_1 + B_{11,1} & 0 & S_2 + \bar{C}_{11,1} \\ S_3 + D_{11,1} & -\bar{C}_{55} & S_1 + B_{11,1} \end{pmatrix}, \tag{34}$$

$$[K_3] = \begin{pmatrix} \bar{C}_{55,1} & 0 & 0 \\ 0 & 0 & 0 \\ -\bar{C}_{55} & 0 & 0 \end{pmatrix}, \{K_4\} = \begin{pmatrix} -\Delta p \\ 0 \\ 0 \end{pmatrix}.$$

A numerical modeling technique, finite difference method (FDM), was applied, as a result, the problem was reduced to solving a system of equations with matrix coefficients. For the resultant algebraic system uses the tridiagonal matrix algorithm (TDMA) [19] in computing the solution with boundary conditions (30) of the form:

$$\begin{cases} [C_0]\{y\}_0 - [B_0]\{y\}_1 = \{F_0\}, \\ -[A_N]\{y\}_{N-1} + [C_N]\{y\}_N = \{F_N\}. \end{cases} \tag{35}$$

7 Calculation results for the stress-strain state of a variable-thickness plate

Consider a model of a variable thickness composite beam made of carbon fibers with the following properties.

$$E_1 = 120\text{GPa}; \mathrm{E}_2 = 3\text{GPa}; \mathrm{E}_3 = 6\text{GPa}, \quad \mu_{12} = 0.1; \mu_{13} = 0.15; \mu_{23} = 0.1,$$
$$G_{12} = 2\text{GPa}; G_{13} = 5\text{GPa}; G_{23} = 2\text{GPa}.$$

The pressure is $\Delta p = 10$Pa, the beam length is $l = \pi$, m, the upper boundary of the beam is given by the function $a^+ = sin(X^1)/5$, and the lower bound is constant $a^- = 0.1$.

To compare the results, we have also considered the solution for a constant thickness beam whose boundaries are the same $a^+ = a^- = 0.1$.

Figure 1 shows the distributions of displacements U_1, deflections W and angles of rotation γ_1 along the length of the beam with variable thickness (red line) and constant (green line).

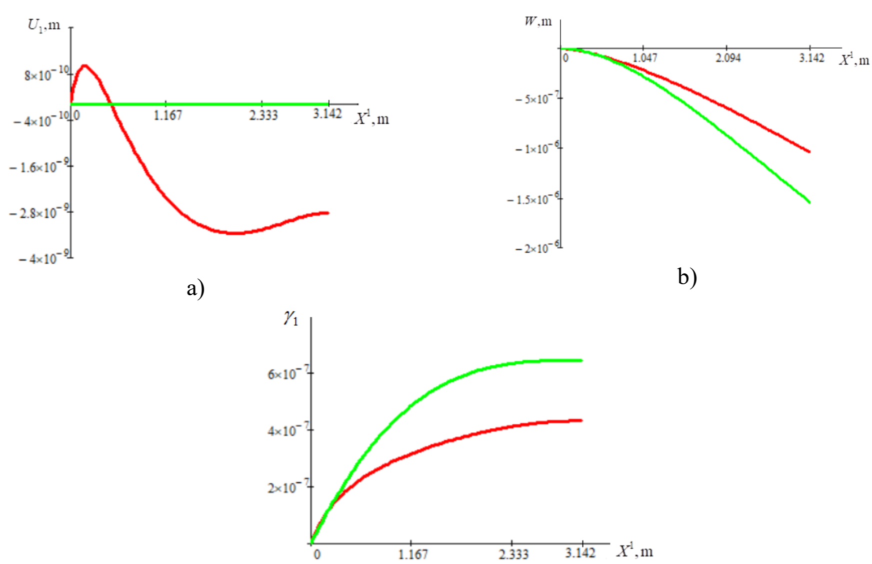

Fig. 1. Distributions of displacement (a), deflection (b), and angle of rotation (c) along the length of the beam for the case of constant (red lines) and variable thickness (green lines).

Figure (2) shows the graphs of the distribution of forces, moments and shear forces for a plate with constant (green line) and variable thickness (red line).

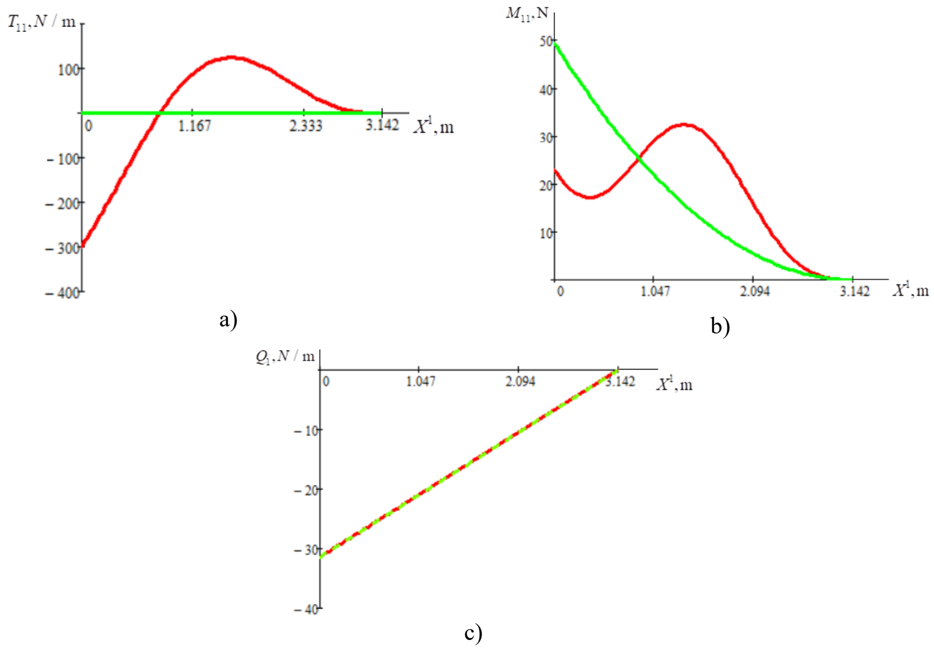

Fig. 2. Distribution of axial force (a), bending moment (b) and shear force (c) along the length of the plate, for the case of constant (green lines) and variable (red lines) thickness.

Figure 2 shows that unlike the forces and moments, the shear force does not depend on the thickness of the beam, and the moment and axial force change significantly. The moment has a maximum value in the middle of the plate and not at the pinched end, as in the case of a constant thickness plate.

8 Conclusions

A model for calculating the stress-strain state of composite shells of variable thickness has been developed, based on assumptions similar to the classical theory of Timoshenko-Mindlin type shells.

As an example, the solution of the problem of plate bending under uniform pressure is considered, taking into account the variable thickness. Comparison of the calculation results with a constant thickness plate shows that the effect of thickness variability is quite significant.

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