

# Multi-row self-balancing device with regard to eccentricity and angular error

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**Abstract.** A model of a self-balancing device is proposed in the article; it is installed with an eccentricity relative to the axis of symmetry of the rotor, and the ball races have a certain horizontal axis of rotation, that is, the balancing masses have an additional degree of freedom. Based on differential equations of motion, the issue of the existence of stationary motions was investigated for various combinations of the installation of a balancing system, with regard to certain eccentricity and angular error, and the rotation of the ball races around the horizontal axis. The numerical solution of the differential equations of motion of the rotor with two balancing balls is obtained and the analysis of the obtained results is conducted. **Keywords:** self-balancing device, rotor, multi-row races, eccentricity, angular error, imbalance.

## 1 Introduction

There is a problem of reducing unwanted vibrations of rotors mounted on an elastic shaft, which occur due to system imbalance in the case of supercritical angular velocities. The article is devoted to the study of the dynamics of the rotor at supercritical angular velocities, with regard to the self-balancing device (SBD). Solving the problem of dynamic balancing of an unbalanced rotor using a balancing device is currently a rather difficult problem since the differential equations of motion are nonlinear and, depending on the model used, have a high order. Therefore, the solution to the problem of dynamic balancing of an unbalanced rotor, for example, using a ball multi-row self-balancing device (SBD) is an urgent problem at present. A sufficient number of scientific publications are devoted to the issues of rotor dynamics with self-balancing devices [1-12].

In these publications, the fundamental results of the dynamics of motion of an unbalanced elastic rotor were obtained with regard to the eccentricity of the center of the self-balancing device, when the race of the balancing balls has a circular or elliptical trajectory. In the case of a circular trajectory, the conditions for the existence of various types of stationary motion with the arrangement of balls along the race are obtained, and the analysis of the obtained

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conditions is conducted. The issues of the stability of particular motions and the passage of the region of critical velocity by the rotor are investigated. In the case of an elliptical race, it is shown that a fully balanced mode is not realized, but there exists a semi-balanced stable stationary motion. It should be noted that in these publications, for specific values of the dynamic and kinematic parameters of the system, numerical results were obtained in the rotor acceleration mode and at a constant value of the torque applied to the rotor axis. The conditions for the existence of this mode were analytically obtained, the question of the stability of the precessional motion of the rotor on elastic supports was investigated, and the analysis of the stability conditions obtained was conducted.

In [8], a full-scale experiment was conducted and an extensive analysis of the dynamics of the SBD was performed, taking into account external damping, friction forces, inertia forces, and eccentricity of the balancing system.

In [13-19], a dynamic unbalanced rotor with various designs of the SBD was considered. In particular, in [15-19] the results of modeling ball balancers installed in a rotor system were considered, taking into account external excitations. It was shown that balancing balls could balance an external force by changing their position; the motion of a multi-mass passive type self-balancing device installed on a rotary machine was studied. Equations of disturbed motion were obtained, and the possibility of depression of equations in independent variables was shown. An analysis of the structure of the system of equations of motion was performed. The features of some special cases were considered, and analytical and numerical studies of the motion of an unbalanced flexible rotating shaft were conducted; the shaft is equipped with balls, the masses of which are distributed along its length. As a result of the study, various models of the SBD for balancing the rotor were proposed. Using the Lagrange method, nonlinear equations of motion were obtained and the balancing effect of the system under consideration was studied within the framework of classical mechanics.

In [20, 21], a mathematical model was developed and the dynamics of various thin-walled elements of elastic structures was studied taking into account the dissipative characteristics of a material with a liquid dynamic absorber. In numerical studies, the expression for the logarithmic decrement of vibrations was determined and the effectiveness of a liquid-type dynamic absorber for damping harmful vibrations of a plate at low frequencies was shown.

A model of a self-contained balancing device was proposed in [22]. The dynamics of the self-balancing device under rotation of the rotor at critical angular velocities was studied. The nonlinear dynamics of the motion of a rotor with two racetracks and two balls was studied numerically at a constant velocity of rotation and in an accelerating mode of motion of the rotor.

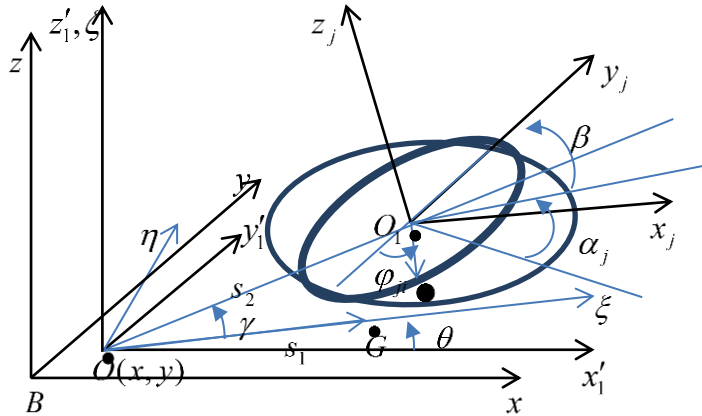
The above is a review of only some of the known publications related to the study of the dynamics of the SBD. According to the formulation and the results obtained in the above studies, one can notice the incompleteness of research on a multi-row SBD, especially when a multi-row self-balancing device has not only an eccentric center but also angular errors.

Therefore, the study devoted to solving these problems is relevant.

## 2 Mathematical formulation of the problem

The self-balancing system is an absolutely rigid rotor in the form of a cylinder mounted on a vertical elastic shaft on two supports. The model considered in [1-4] is taken as a mathematical model, with the addition that circular race lines (tubes) can perform rotational motion around one specific horizontal axis. It is assumed that the rotor is a cylinder of small height and performs a plane motion (in the framework of the Jeffcott model). The distance between the geometric center  $O$  and the center of gravity  $G$  of the cylinder, i.e., static eccentricity, is denoted by  $s_1$ . There is a balancing device to eliminate the imbalance of the

cylinder in the form of circular tubes with  $n$  balls of various masses, centered at point  $O_1$ . We introduce the parameters  $s_2 = OO_1$  and the angle of  $\gamma = \angle O_1OG$  between the directions  $OO_1$  and  $OG$ , which characterize the eccentricity of the center of the balancing system. We introduce the angle  $\beta$  between the horizontal axis of rotation of the tube and the direction  $OO_1$  (Fig.1). For the convenience of comparing expressions, the main notation is taken as in [1].



**Fig. 1.** Rotor with balancing system.

To describe the state of the system, we introduce a fixed coordinate system  $Bxyz$  with the  $Bz$  - axis parallel to the rotation axis passing through the support points. The  $Bx, By$  axes lie in the plane of static eccentricity. A moving coordinate system  $O\xi\eta\zeta$  is also introduced, located at the fixing point and rotating with the rotor. In this case, the direction of the  $O\xi$  -axis coincides with the  $Bz$  -axis. To determine the relative motion of the ball races and balancing masses, we introduce moving coordinate systems  $O_1x_jy_jz_j$  connected with tubes, the beginning of which is located in the center of the balancing system  $O_1$ .  $O_1y_j$  -axes are directed along the horizontal axis of rotation of the tubes, and the  $O_1x_j$  and  $O_1z_j$  axes form the right-hand coordinate system (Fig. 1).

The degree of freedom of the system (Fig. 1) is  $k = s \times (n + 1) + 3$ . Thus,  $x, y$  are the coordinates of the center of mass of the rotor,  $\theta$  is the angle of rotation of the rotor around the vertical axis,  $\alpha_j (j = 1, \dots, s)$  are the angles of the greatest inclination of the plane of the tubes to the  $Bxy$  plane. To determine the position of the balls inside the tube, we introduce the angles  $\varphi_{ji} (i = 1, \dots, n; j = 1, \dots, s)$  between the  $O_1y_j$  -axis passing through the center of face circles, and the radii drawn from the center of circle to the balls.

The mechanical system has geometric constraints. Lagrange's equations in generalized coordinates are used to derive differential equations of motion.

The kinetic energy of the system can be written in the following form:

$$T = \frac{1}{2} m_p \bar{v}_G^2 + \frac{1}{2} J_G \dot{\theta}^2 + \frac{1}{2} m_{\bar{oc}} \bar{v}_{O_1}^2 + \frac{1}{2} \sum_{j=1}^s (J_{y_j} \dot{\alpha}_j^2 + J_{x_{j2}} \dot{\theta}^2 \sin^2 \alpha_j + J_{z_{j2}} \dot{\theta}^2 \cos^2 \alpha_j) + \\ + \frac{1}{2} \sum_{j=1}^s \sum_{i=1}^n m_{ji} \bar{v}_{ji}^2,$$

Here,  $\bar{v}_G$  is the velocity of the center of mass of the cylinder;  $\bar{v}_{O_1}$  is the velocity of the center of the balancing mechanism;  $\bar{v}_{ji}$  is the velocity of the ball located in the tube with serial number  $j$ ;  $m_p, m_{\bar{oc}}, m_{ji}$  are the mass of the rotor, the mass of the balancing device and the mass of the balls, respectively;  $J_G$  is the moment of inertia of the rotor relative to the main axis of the rotor;  $J_{y_{j2}}, J_{x_{j2}}, J_{z_{j2}}$  are the moments of inertia of the tubes relative to the main axes of inertia.

The introduced values are defined as follows:

$$\bar{v}_G = (\dot{x} - s_1 \sin \theta \dot{\theta}) \vec{i} + (\dot{y} + s_1 \cos \theta \dot{\theta}) \vec{j};$$

$$\bar{v}_{O_1} = (\dot{x} - s_2 \sin(\theta + \gamma) \dot{\theta}) \vec{i} + (\dot{y} + s_2 \cos(\theta + \gamma) \dot{\theta}) \vec{j};$$

$$\bar{v}_{ji} = (\dot{x} + (-s_2 \sin(\theta + \beta) + r_j \cos \varphi_{ji} \sin \delta + r_j \sin \varphi_{ji} \cos \delta \cos \alpha_j) \dot{\theta} - r_j \sin \varphi_{ji} \sin \alpha_j \sin \delta \dot{\alpha}_j + \\ + (\sin \varphi_{ji} \cos \delta + \cos \varphi_{ji} \cos \alpha_j \sin \delta) r_j \dot{\varphi}_{ji}) \vec{i} + (\dot{y} + (s_2 \cos(\theta + \beta) - r_j \cos \varphi_{ji} \cos \delta + \\ + r_j \sin \varphi_{ji} \cos \alpha_j \sin \delta) \dot{\theta} + r_j \sin \varphi_{ji} \sin \alpha_j \cos \delta \dot{\alpha}_j + r_j \dot{\varphi}_{ji} (\sin \varphi_{ji} \sin \delta - \cos \varphi_{ji} \cos \alpha_j \cos \delta)) \vec{j} + \\ + (r_j \sin \varphi_{ji} \cos \alpha_j \dot{\alpha}_j + r_j \dot{\varphi}_{ji} \cos \varphi_{ji} \sin \alpha_j) \vec{k}, \quad \delta = \theta + \gamma + \beta, \quad (j=1, \dots, s; \quad i=1, \dots, n).$$

Therefore, the kinetic energy of the system in explicit form in generalized coordinates has the following form:

$$T = \frac{1}{2} (m_p + m_{\bar{oc}} + \sum_{j=1}^s \sum_{i=1}^n m_{ji}) (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} \sum_{j=1}^s (J_G + J_{z_{j2}} \cos^2 \alpha_j + J_{x_{j2}} \sin^2 \alpha_j + m_p s_1^2 + m_{\bar{oc}} s_2^2) \dot{\theta}^2 + \\ + \sum_{j=1}^s \sum_{i=1}^n m_{ji} (s_2^2 + 2r_j s_2 (\sin \varphi_{ji} \cos \alpha_j \sin \beta - \cos \varphi_{ji} \cos \beta) + r_j^2 (\cos^2 \varphi_{ji} + \sin^2 \varphi_{ji} \cos^2 \alpha_j)) \dot{\theta}^2 + \\ + \frac{1}{2} \sum_{j=1}^s (J_{y_{j2}} + \sum_{i=1}^n m_{ji} r_j^2 \sin^2 \varphi_{ji}) \dot{\alpha}_j^2 + \frac{1}{2} \sum_{j=1}^s \sum_{i=1}^n m_{ji} r_j^2 \dot{\varphi}_{ji}^2 + m_p s_1 \dot{\theta} (\dot{y} \cos \theta - \dot{x} \sin \theta) + \\ + m_{\bar{oc}} s_2 \dot{\theta} (\dot{y} \cos(\theta + \gamma) - \dot{x} \sin(\theta + \gamma)) + \sum_{j=1}^s \sum_{i=1}^n m_{ji} [\dot{x} (-s_2 \sin(\gamma + \theta) \dot{\theta} + r_j (\dot{\theta} \cos \alpha_j + \dot{\varphi}_{ji}) \sin \varphi_{ji} \cos \delta + \\ + r_j (\dot{\theta} + \dot{\varphi}_{ji} \cos \alpha_j) \cos \varphi_{ji} \sin \delta - r_j \sin \varphi_{ji} \sin \alpha_j \sin \delta \dot{\alpha}_j) + \dot{y} (s_2 \cos(\theta + \gamma) \dot{\theta} + \\ + r_j (\dot{\theta} \cos \alpha_j + \dot{\varphi}_{ji}) \sin \varphi_{ji} \sin \delta - r_j (\dot{\theta} + \dot{\varphi}_{ji} \cos \alpha_j) \cos \varphi_{ji} \cos \delta + r_j \sin \varphi_{ji} \sin \alpha_j \cos \delta \dot{\alpha}_j) + \\ + ((\sin \varphi_{ji} \sin \beta - \cos \varphi_{ji} \cos \alpha_j \cos \beta) s_2 r_j + r_j^2 \cos \alpha_j) \dot{\theta} \dot{\varphi}_{ji} + (s_2 r_j \cos \beta \sin \varphi_{ji} \sin \alpha_j - \\ - r_j^2 \sin \varphi_{ji} \cos \varphi_{ji} \sin \alpha_j) \dot{\theta} \dot{\alpha}_j].$$

$$\Pi = \frac{1}{2}k(x^2 + y^2)$$

If we take into account the potential energy of the elastic shaft and the external dissipative function in all variables, then we have:

$$F = \frac{1}{2}[c_\theta \dot{\theta}^2 + c(\dot{x}^2 + \dot{y}^2) + c_\alpha \sum_{j=1}^s \dot{\alpha}_j^2 + c_\varphi \sum_{j=1}^s \sum_{i=1}^n \dot{\varphi}_{ji}^2].$$

In generalized coordinates, the equation of motion is written in the following form:

$$\begin{aligned} M\ddot{x} + c\dot{x} + kx &= [m_p s_1 \cos \theta + (m_{oc} + \sum_{j=1}^s \sum_{i=1}^n m_{ji}) s_2 \cos(\theta + \gamma) - \sum_{j=1}^s \sum_{i=1}^n m_{ji} r_j (\sin \varphi_{ji} \cos \alpha_j \sin \delta - \\ &- \cos \varphi_{ji} \cos \delta)] \dot{\theta}^2 + [m_p s_1 \sin \theta + (m_{oc} + \sum_{j=1}^s \sum_{i=1}^n m_{ji}) s_2 \sin(\theta + \gamma) + \sum_{j=1}^s \sum_{i=1}^n m_{ji} r_j \sin \varphi_{ji} \cos \alpha_j \cos \delta + \\ &+ \cos \varphi_{ji} \sin \delta] \ddot{\theta} + \sum_{j=1}^s \sum_{i=1}^n m_{ji} r_j [(\cos \varphi_{ji} \cos \delta \cos \alpha_j - \sin \varphi_{ji} \sin \delta) \dot{\varphi}_{ji} - \sin \varphi_{ji} \sin \alpha_j \cos \delta \dot{\alpha}_j] \dot{\theta} - \\ &- \sum_{j=1}^s \sum_{i=1}^n m_{ji} r_j \frac{d}{dt} [(\sin \varphi_{ji} \cos \delta + \cos \varphi_{ji} \cos \alpha_j \sin \delta) \dot{\varphi}_{ji} - \sin \varphi_{ji} \sin \alpha_j \sin \delta \dot{\alpha}_j], \\ M\ddot{y} + c\dot{y} + ky &= [m_p s_1 \sin \theta + (m_{oc} + \sum_{j=1}^s \sum_{i=1}^n m_{ji}) s_2 \sin(\theta + \gamma) - \sum_{j=1}^s \sum_{i=1}^n m_{ji} r_j (\sin \varphi_{ji} \cos \alpha_j \cos \delta + \\ &+ \cos \varphi_{ji} \sin \delta)] \dot{\theta}^2 - [m_p s_1 \cos \theta + (m_{oc} + \sum_{j=1}^s \sum_{i=1}^n m_{ji}) s_2 \cos(\theta + \gamma) + \sum_{j=1}^s \sum_{i=1}^n m_{ji} r_j \sin \varphi_{ji} \cos \alpha_j \sin \delta - \\ &+ \cos \varphi_{ji} \cos \delta] \ddot{\theta} - \sum_{j=1}^s \sum_{i=1}^n m_{ji} r_j [(\cos \varphi_{ji} \cos \alpha_j \sin \delta + \sin \varphi_{ji} \sin \delta) \dot{\varphi}_{ji} - \sin \varphi_{ji} \sin \alpha_j \sin \delta \dot{\alpha}_j] \dot{\theta} - \\ &- \sum_{j=1}^s \sum_{i=1}^n m_{ji} r_j \frac{d}{dt} [(\sin \varphi_{ji} \sin \delta - \cos \varphi_{ji} \cos \alpha_j \cos \delta) \dot{\varphi}_{ji} + \sin \varphi_{ji} \sin \alpha_j \cos \delta \dot{\alpha}_j], \\ m_{ji} r_j^2 \ddot{\varphi}_{ji} + m_{ji} r_j \sin \varphi_{ji} (\ddot{x} \cos \delta + \ddot{y} \sin \delta) + m_{ji} r_j \cos \varphi_{ji} \cos \alpha_j (\ddot{x} \sin \delta - \ddot{y} \cos \delta) + \\ &+ m_{ji} \frac{d}{dt} [(\sin \varphi_{ji} \sin \beta - \cos \varphi_{ji} \cos \alpha_j \cos \beta) s_2 r_j + r_j^2 \cos \alpha_j] \dot{\theta} - [m_{ji} r_j s_2 (\cos \varphi_{ji} \sin \beta + \\ &+ \sin \varphi_{ji} \cos \alpha_j \cos \beta) \dot{\theta} \dot{\varphi}_{ji} + m_{ji} (r_j s_2 \cos \varphi_{ji} \sin \alpha_j \cos \beta - r_j^2 \sin 2\varphi_{ji} \sin \alpha_j) \dot{\theta} \dot{\alpha}_j + \\ &+ m_{ji} r_j s_2 (\sin \varphi_{ji} \cos \beta + \cos \varphi_{ji} \cos \alpha_j \sin \beta)] \dot{\theta}^2 - m_{ji} r_j^2 \sin^2 \alpha_j \sin 2\varphi_{ji} \dot{\theta}^2 - \\ &- m_{ji} r_j^2 \sin \varphi_{ji} \cos \varphi_{ji} \dot{\alpha}_j^2 = -c_\varphi \dot{\varphi}_{ji} - m_{ji} g r_j \cos \varphi_{ji} \sin \alpha_j, (i = 1, \dots, n) \\ \sum_{i=1}^n [(m_{ji} r_j^2 \sin^2 \varphi_{ji} + J_{x_2}) \ddot{\alpha}_j - m_{ji} (\ddot{x} \sin \delta - \ddot{y} \cos \delta) \sin \varphi_{ji} \sin \alpha_j + m_{ji} \frac{d}{dt} [s_2 r_j \sin \varphi_{ji} \sin \alpha_j \cos \beta - \\ &- r_j^2 \sin \varphi_{ji} \cos \varphi_{ji} \sin \alpha_j] \dot{\theta}] + 2m_{ji} r_j^2 \sin \varphi_{ji} \cos \varphi_{ji} \dot{\alpha}_j \dot{\varphi}_{ji} - m_{ji} [(s_2 r_j \cos \varphi_{ji} \sin \alpha_j \cos \beta - r_j^2 \sin \alpha_j) \dot{\theta} \dot{\varphi}_{ji} + \\ &+ (s_2 r_j \cos \beta \sin \varphi_{ji} - r_j^2 \sin \varphi_{ji}) \cos \alpha_j \dot{\theta} \dot{\alpha}_j] - [\frac{1}{2} (J_{x_2} - J_{z_2}) \sin 2\alpha_j + m_{ji} (-s_2 r_j \sin \beta \sin \varphi_{ji} \sin \alpha_j) - \\ &- r_j^2 \sin^2 \varphi_{ji} \sin \alpha_j \cos \alpha_j] \dot{\theta}^2 = -c_\alpha \dot{\alpha}_j - m_{ji} g r_j \sin \varphi_{ji} \cos \alpha_j, (j = 1, \dots, s) \end{aligned}$$

$$\begin{aligned}
& (m_p s_1^2 + m_{oc} s_2^2 + \sum_{j=1}^s \sum_{i=1}^n m_{ji} s_2^2 + J_G + \sum_{j=1}^n (J_{x_j} \sin^2 \alpha_j + J_{z_j} \cos^2 \alpha_j)) \ddot{\theta} + c_\theta \dot{\theta} = m_p s_1 (\sin \theta \ddot{x} - \cos \theta \ddot{y}) + \\
& + m_{oc} s_2 (\sin(\theta + \gamma) \ddot{x} - \cos(\theta + \gamma) \ddot{y}) - \sum_{j=1}^s \sum_{i=1}^n m_{ji} (r_j \sin \varphi_{ji} \cos \alpha_j (\cos \delta \ddot{x} - \sin \delta \ddot{y}) + r_j \cos \varphi_{ji} (\sin \delta \ddot{x} - \\
& - \cos \delta \ddot{y}) - \frac{d}{dt} [(s_2^2 + r_j^2 \sin^2 \varphi_{ji} \cos^2 \alpha_j + r_j^2 \cos^2 \varphi_{ji} + 2s_2 r_j \sin \varphi_{ji} \cos \alpha_j \sin \beta - 2s_2 r_j \cos \varphi_{ji} \cos \beta) \dot{\theta} + \\
& + (s_2 r_j (\sin \varphi_{ji} \sin \beta - \cos \varphi_{ji} \cos \alpha_j \cos \beta) + r_j^2 \cos \alpha_j) \dot{\varphi}_{ji} + (s_2 r_j \sin \varphi_{ji} \sin \alpha_j \cos \beta - \\
& - r_j^2 \sin \varphi_{ji} \cos \varphi_{ji} \sin \alpha_j) \dot{\alpha}_j]) + M_\theta.
\end{aligned}$$

$$\text{where } M = (m_p + m_{oc} + \sum_{j=1}^s \sum_{i=1}^n m_{ji}).$$

Often, when studying rotor systems, it is appropriate to use a rotating coordinate system related to the rotor. To transfer to a moving coordinate system, we use the standard change of variables. Instead of variables  $x, y, \theta$ , we introduce new variables according to the following formulas

$$z = x + iy = \zeta e^{i\theta}, \dot{x} + i\dot{y} = (\dot{\zeta} + \zeta \dot{\theta} i) e^{i\theta}, \ddot{x} + i\ddot{y} = (\ddot{\zeta} - \zeta \dot{\theta}^2 + (2\dot{\zeta} \dot{\theta} + \zeta \ddot{\theta}) i) e^{i\theta}. \quad (2)$$

Substituting these variables into the main system (1), we obtain the following self-contained system of differential equations of motion with respect to variable:  $\zeta, \alpha_j, \varphi_{ji}$

$$\begin{aligned}
& [M(\ddot{\zeta} - \zeta \dot{\theta}^2 + i(2\dot{\zeta} \dot{\theta} + \zeta \ddot{\theta})) + c(\dot{\zeta} + i\zeta \dot{\theta}) + k\zeta] e^{i\theta} = \frac{d}{dt} \{ [-m_p s_1 i - (m_{oc} + \sum_{j=1}^s \sum_{i=1}^n m_{ij}) i s_2] e^{i\gamma} - \\
& - \sum_{j=1}^s \sum_{i=1}^n m_{ji} r_j (-i \cos \varphi_{ji} e^{i(\gamma+\beta)} + \sin \varphi_{ji} \cos \alpha_j e^{i(\beta+\gamma)}) \dot{\theta} - \\
& - \sum_{j=1}^s \sum_{i=1}^n m_{ji} r_j [(\sin \varphi_{ji} e^{i(\beta+\gamma)} - i \cos \varphi_{ji} \cos \alpha_j e^{i(\beta+\gamma)}) \dot{\varphi}_{ji} + i \sin \varphi_{ji} \sin \alpha_j e^{i(\beta+\gamma)} \dot{\alpha}_j] \} e^{i\theta},
\end{aligned} \quad (3)$$

$$\begin{aligned}
& m_{ji} r_j^2 \ddot{\varphi}_{ji} + m_{ji} r_j \sin \varphi_{ji} (\operatorname{Re}(\ddot{z} e^{-i\delta}) + m_{ji} r_j \cos \varphi_{ji} \cos \alpha_j \operatorname{Im}(\ddot{z} e^{-i\delta})) + \\
& + m_{ji} \frac{d}{dt} [(\sin \varphi_{ji} \sin \beta - \cos \varphi_{ji} \cos \alpha_j \cos \beta) s_2 r_j + r_j^2 \cos \alpha_j] \dot{\theta} - [m_{ji} r_j s_2 (\cos \varphi_{ji} \sin \beta + \\
& + \sin \varphi_{ji} \cos \alpha_j \cos \beta) \dot{\theta} \dot{\varphi}_{ji} + m_{ji} (r_j s_2 \cos \varphi_{ji} \sin \alpha_j \cos \beta - r_j^2 \sin 2\varphi_{ji} \sin \alpha_j) \dot{\theta} \dot{\alpha}_j + \\
& + m_{ji} r_j s_2 (\sin \varphi_{ji} \cos \beta + \cos \varphi_{ji} \cos \alpha_j \sin \beta) \dot{\theta}^2 - m_{ji} r_j^2 \sin^2 \alpha_j \sin 2\varphi_{ji} \dot{\theta}^2] - \\
& - m_{ji} r_j^2 \sin \varphi_{ji} \cos \varphi_{ji} \dot{\alpha}_j^2 = -c_\varphi \dot{\varphi}_{ji} - m_{ji} g r_j \cos \varphi_{ji} \sin \alpha_j, (i = 1, \dots, n)
\end{aligned}$$

$$\begin{aligned} & \sum_{i=1}^n [(m_{ji}r_j^2 \sin^2 \varphi_{ji} + J_{y_2})\ddot{\alpha}_j - m_{ji} \operatorname{Im}(\ddot{z}e^{-i\delta}) \sin \varphi_{ji} \sin \alpha_j + m_{ji} \frac{d}{dt} [s_2 r_j \sin \varphi_{ji} \sin \alpha_j \cos \beta - \\ & - r_j^2 \sin \varphi_{ji} \cos \varphi_{ji} \sin \alpha_j] \dot{\theta}] + 2m_{ji}r_j^2 \sin \varphi_{ji} \cos \varphi_{ji} \dot{\alpha}_j \dot{\varphi}_{ji} - m_{ji} [(s_2 r_j \cos \varphi_{ji} \sin \alpha_j \cos \beta - r_j^2 \sin \alpha_j) \dot{\theta} \dot{\varphi}_{ji} + \\ & + (s_2 r_j \cos \beta \sin \varphi_{ji} - r_j^2 \sin \varphi_{ji}) \cos \alpha_j \dot{\theta} \dot{\alpha}_j] - [\frac{1}{2} (J_{x_2} - J_{z_2}) \sin 2\alpha_j - m_{ji} (s_2 r_j \sin \beta \sin \varphi_{ji} \sin \alpha_j + \\ & + r_j^2 \sin^2 \varphi_{ji} \sin \alpha_j \cos \alpha_j)] \dot{\theta}^2] = -c_\alpha \dot{\alpha}_j - m_{ji} g r_j \sin \varphi_{ji} \cos \alpha_j, (j = 1, \dots, s) \\ & (m_p s_1^2 + m_{\theta c} s_2^2 + \sum_{j=1}^s \sum_{i=1}^n m_{ji} s_2^2 + J_G + \sum_{j=1}^n (J_{x_j} \sin^2 \alpha_j + J_{z_j} \cos^2 \alpha_j)) \ddot{\theta} + c_\theta \dot{\theta} = m_p s_1 \operatorname{Im}(\ddot{z}e^{-i\theta}) - \\ & + m_{\theta c} s_2 \operatorname{Im}(\ddot{z}e^{-i(\theta+\gamma)}) - \sum_{j=1}^s \sum_{i=1}^n m_{ji} (r_j \sin \varphi_{ji} \cos \alpha_j \operatorname{Re}(\ddot{z}e^{i\theta}) - r_j \cos \varphi_{ji} \operatorname{Im}(\ddot{z}e^{-i\delta}) - \\ & - \frac{d}{dt} [(s_2^2 + r_j^2 \sin^2 \varphi_{ji} \cos^2 \alpha_j + r_j^2 \cos^2 \varphi_{ji} + 2s_2 r_j \sin \varphi_{ji} \cos \alpha_j \sin \beta - 2s_2 r_j \cos \varphi_{ji} \cos \beta) \dot{\theta} + \\ & + (s_2 r_j (\sin \varphi_{ji} \sin \beta - \cos \varphi_{ji} \cos \alpha_j \cos \beta) + r_j^2 \cos \alpha_j) \dot{\varphi}_{ji} + (s_2 r_j \sin \varphi_{ji} \sin \alpha_j \cos \beta - \\ & - r_j^2 \sin \varphi_{ji} \cos \varphi_{ji} \sin \alpha_j) \dot{\alpha}_j]) + M_\theta. \end{aligned}$$

### 3 Stationary motion

Below we consider stationary motions at constant angular velocity  $\dot{\theta} = \nu$ , that is, partial solutions are sought in the following form

$$\zeta = A e^{i\phi_0}, \dot{\theta} = \nu, \varphi_{ji} = \varphi_{ji}^0 = \text{const}, A = \text{const}, \phi_0 = \text{const}, \alpha_j = \alpha_j^0 = \text{const}. \quad (4)$$

Where  $A$  and  $\phi_0$  are the constant amplitude and phase of the rotational shear.

The necessary conditions for the existence of stationary motion can be obtained by substituting (4) into (3), i.e.:

$$\begin{aligned} & [((k - M\nu^2)\xi_0 - c\nu\eta_0) + ((k - M\nu^2)\eta_0 + c\nu\xi_0)i] = [m_p s_1 + (m_{\theta c} + \sum_{j=1}^s \sum_{i=1}^n m_{ji}) s_2 e^{i\gamma} - \\ & - \sum_{j=1}^s \sum_{i=1}^n m_{ji} r_j (\cos \varphi_{ji}^0 e^{i(\gamma+\beta)} - i \sin \varphi_{ji}^0 \cos \alpha_j^0 e^{i(\gamma+\beta)})] \nu^2, \end{aligned} \quad (5)$$

$$\begin{aligned} & \sin \varphi_{ji}^0 (\operatorname{Re}(\ddot{z}e^{-i\delta}) + \cos \varphi_{ji}^0 \cos \alpha_j^0 \operatorname{Im}(\ddot{z}e^{-i\delta})) + (s_2 (\sin \varphi_{ji}^0 \cos \beta + \cos \varphi_{ji}^0 \cos \alpha_j^0 \sin \beta) - \\ & - r_j \sin^2 \alpha_j^0 \sin 2\varphi_{ji}^0) = -\frac{g}{\nu^2} \cos \varphi_{ji}^0 \sin \alpha_j^0, (i = 1, \dots, n) \end{aligned}$$

$$\begin{aligned} & -m_{ji} r_j \operatorname{Im}(\ddot{z}e^{-i\delta}) \sin \varphi_{ji}^0 \sin \alpha_j^0 - \frac{1}{2} (J_{x_2} - J_{z_2}) \sin 2\alpha_j^0 + m_{ji} (s_2 r_j \sin \beta \sin \varphi_{ji}^0 \sin \alpha_j^0 + \\ & + r_j^2 \sin^2 \varphi_{ji}^0 \sin \alpha_j^0 \cos \alpha_j^0) = -m_{ji} \frac{g}{\nu^2} r_j \sin \varphi_{ji}^0 \cos \alpha_j^0. (j = 1, \dots, s) \end{aligned}$$

Thus, (5) are the conditions under which stationary motions of the form (4) hold.

In the general case, finding an analytical solution to system (5) is a rather difficult task. Therefore, for specific values of the system parameters, for the case when there are two ball

races and one ball in each of them, the solution was obtained by a numerical method. For the following values of system parameters

$$m_p = 1.5 \text{ kg}; m_{\delta c} = 0.7 \text{ kg}; m_{11} = 0.05 \text{ kg}; m_{21} = 0.06 \text{ kg}; r_1 = 0.03 \text{ m}; r_2 = 0.025 \text{ m};$$

$$\beta = \frac{\pi}{6} \text{ rad}; \nu = 600 \text{ rad / s};$$

numerical method gives the following values:

$$\xi_0 = 5 \cdot 10^{-11} \text{ m}, \eta_0 = 4.5 \cdot 10^{-11} \text{ m}, \varphi_{11} = -4.7 \text{ rad}, \varphi_{21} = -11.5 \text{ rad},$$

$$\alpha_1 = -9.7 \cdot 10^{-8} \text{ rad}, \alpha_2 = 3.142668 \cdot 10^{-8} \text{ rad}$$

It can be seen that the balls of races are located almost horizontally, the balancing balls are located approximately opposite to each other, and the rotor performs an unbalanced rotational motion with a constant angular velocity.

### 3.1. Special cases

1. In the SBD system, all ball races are located in the same plane at a small angle of  $\Delta$  relative to the horizontal plane, that is, not only eccentricity is considered, but also a small angular error of mounting.

Assuming that in system (5)  $\alpha_1^0 = \alpha_2^0 = \dots = \alpha_s^0 = \alpha_0$  and  $\sin \alpha_0 \approx \alpha_0, \cos \alpha_0 \approx 1$ , we obtain

$$\begin{aligned} \left(\frac{k}{\nu^2} - M\right)\xi_0 - \frac{c}{\nu}\eta_0 &= m_p s_1 + \sum_{j=1}^s \sum_{i=1}^n (m_{\delta c} + m_{ji}) s_2 \cos \gamma - \sum_{j=1}^s \sum_{i=1}^n m_{ji} r_j \cos(\varphi_{ji}^0 + \Delta), \\ \frac{c}{\nu}\xi_0 - \left(\frac{k}{\nu^2} - M\right)\eta_0 &= \sum_{j=1}^s \sum_{i=1}^n (m_{\delta c} + m_{ji}) s_2 \sin \gamma - \sum_{j=1}^s \sum_{i=1}^n m_{ji} r_j \sin(\varphi_{ji}^0 + \Delta), \quad (\Delta = \beta + \gamma) \end{aligned} \quad (6)$$

$$\xi_0 \sin(\varphi_{ji}^0 + \Delta) - \eta_0 \cos(\varphi_{ji}^0 + \Delta) + s_2 \sin(\varphi_{ji}^0 + \beta) = \frac{g}{\nu^2} \cos \varphi_{ji}^0 \alpha_0, \quad (i = 1, 2, \dots, n; j = 1, 2, \dots, s).$$

For  $i = 1, j = 2$  ( $i$  is the number of balls,  $j$  is the number of races), that is, there is one ball in each race, we have

$$\begin{aligned} \left(\frac{k}{\nu^2} - M\right)\xi_0 - \frac{c}{\nu}\eta_0 &= m_p s_1 + (m_{\delta c} + m_{11} + m_{21}) s_2 \cos \gamma - (m_{11} r_1 \cos(\varphi_{11}^0 + \Delta) + m_{21} r_2 \cos(\varphi_{21}^0 + \Delta)), \\ \frac{c}{\nu}\xi_0 - \left(\frac{k}{\nu^2} - M\right)\eta_0 &= (m_{\delta c} + m_{11} + m_{21}) s_2 \sin \gamma - (m_{11} r_1 \sin(\varphi_{11}^0 + \Delta) + m_{21} r_2 \sin(\varphi_{21}^0 + \Delta)), \end{aligned} \quad (7)$$



$$\xi_0 \sin(\varphi_{11}^0 + \Delta) - \eta_0 \cos(\varphi_{11}^0 + \Delta) + s_2 \sin(\varphi_{11}^0 + \beta) = \frac{\bar{g}}{v^2} \cos \varphi_{11}^0 \alpha_0,$$

$$\xi_0 \sin(\varphi_{21}^0 + \Delta) - \eta_0 \cos(\varphi_{21}^0 + \Delta) + s_2 \sin(\varphi_{21}^0 + \beta) = \frac{\bar{g}}{v^2} \cos \varphi_{21}^0 \alpha_0, (\bar{g} = \frac{\bar{g}}{v^2})$$

The last two relations in (7) after some transformations have the following form:

$$\sin(\varphi_{11}^0 - \varphi_{21}^0)(\eta_0 + \bar{g} \alpha \cos \Delta + s_2 \sin \gamma) = 0,$$

$$\sin(\varphi_{11}^0 - \varphi_{21}^0)(\xi_0 - \bar{g} \alpha \sin \Delta + s_2 \cos \gamma) = 0.$$

Here

$$\sin(\varphi_{11}^0 - \varphi_{21}^0) = 0, \quad (8)$$

$$\text{or } \xi_0 = -s_2 \cos \gamma + \bar{g} \alpha_0 \sin \Delta, \quad \eta_0 = -s_2 \sin \gamma - \bar{g} \alpha_0 \cos \Delta. \quad (9)$$

From (8) we obtain  $\varphi_{11}^0 - \varphi_{21}^0 = k\pi$ , which coincides with the result of the arrangement of the balls given in [1-2], i.e.,  $\varphi_{11} = \varphi_{21} = 0$ ,  $\varphi_{11} = 0$ ,  $\varphi_{21} = \pi$  and  $\varphi_{11} = \pi$ ,  $\varphi_{21} = \pi$ .

In the second case, there is an unbalanced rotational motion of the rotor with a displacement of the geometric center by the value of

$$\xi_0 = -s_2 \cos \gamma + \bar{g} \alpha_0 \sin \Delta,$$

$$\eta_0 = -s_2 \sin \gamma - \bar{g} \alpha_0 \cos \Delta.$$

To determine the location of the balls along the race, we substitute (9) into the first two equations in (7).

$$m_{11} r_1 \cos(\varphi_{11}^0 + \Delta) + m_{21} r_2 \cos(\varphi_{21}^0 + \Delta) = \left[ \left( \frac{k}{v^2} - M \right) \xi_0 - \frac{c}{v} \eta_0 \right] - m_p s_1 - (m_{\delta c} + m_{11} + m_{21}) s_2 \cos \gamma, \quad (10)$$

$$m_{11} r_1 \sin(\varphi_{11}^0 + \Delta) + m_{21} r_2 \sin(\varphi_{21}^0 + \Delta) = \left[ \frac{c}{v} \xi_0 - \left( \frac{k}{v^2} - M \right) \eta_0 \right] - (m_{\delta c} + m_{11} + m_{21}) s_2 \sin \gamma,$$

If  $m_{11} r_1 = m_{21} r_2$ , then from (10) we obtain the following values for angles  $\varphi_{11}^0, \varphi_{21}^0$

$$\cos(\varphi_{11}^0 - \varphi_{21}^0) = \frac{A_1^2 + A_2^2 - 2}{2}, \quad (11)$$

$$\operatorname{tg} \frac{\varphi_{11}^0 + \varphi_{21}^0 + 2\Delta}{2} = \frac{A_1}{A_2}.$$

Where

$$A_1 = \left( \left[ \left( \frac{k}{v^2} - M \right) \xi_0 - \frac{c}{v} \eta_0 \right] - m_p s_1 - (m_{\delta c} + m_{11} + m_{21}) s_2 \cos \gamma \right) / m_{11} r_1,$$

$$A_2 = \left( \left[ \frac{c}{v} \xi_0 - \left( \frac{k}{v^2} - M \right) \eta_0 \right] - (m_{\delta c} + m_{11} + m_{21}) s_2 \sin \gamma \right) / m_{21} r_2.$$

Thus, the position of the balls in the races is determined from relations (11), as in [1], with an offset by an angle  $\Delta = \gamma + \beta$ .

2. For  $A = 0$ , we have a balanced stationary mode. Substituting  $\xi_0 = 0, \eta_0 = 0$  into (6), we obtain:

$$m_p s_1 + \sum_{j=1}^s \sum_{i=1}^n (m_{\delta c} + m_{ji}) s_2 \cos \gamma - \sum_{j=1}^s \sum_{i=1}^n m_{ji} r_j \cos(\varphi_{ji}^0 + \Delta) = 0,$$

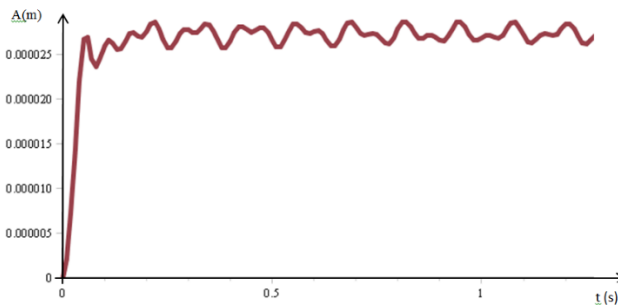
$$\sum_{j=1}^s \sum_{i=1}^n (m_{\delta c} + m_{ji}) s_2 \sin \gamma - \sum_{j=1}^s \sum_{i=1}^n m_{ji} r_j \sin(\varphi_{ji}^0 + \Delta) = 0, \quad (\Delta = \beta + \gamma)$$

In the general case, the system of equations does not hold. But there are special cases that coincide with the results given in [1] with an angular accuracy of  $\Delta = \gamma + \beta$ .

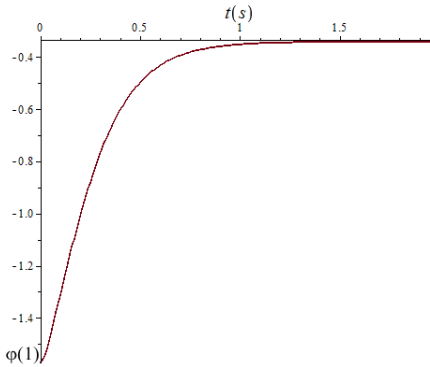
It should be noted that, in the general case, finding an analytical solution to the system of equations (3) is a rather difficult task. Therefore, equation (3) below is solved numerically for specific values of the system parameters. Figure 2 shows the numerical results obtained over time, for the geometric center of the rotor with SBD and the position of the balls for the following parameters (there is one ball in each race):

$$m_p = 1.5 \text{ kg}; m_{\delta c} = 0.7 \text{ kg}; m_{11} = 0.05 \text{ kg}; m_{21} = 0.06 \text{ kg}; r_1 = 0.03 \text{ m}; r_2 = 0.025 \text{ m};$$

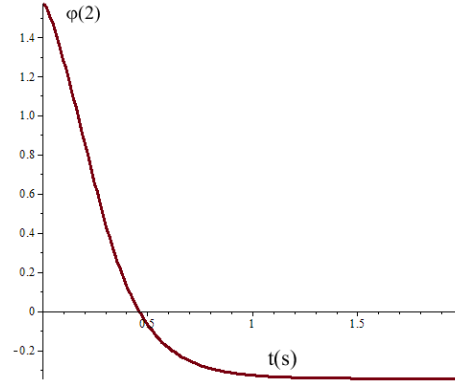
$$\alpha_1 = \alpha_2 = \frac{\pi}{40} \text{ rad}; \beta = \frac{\pi}{6} \text{ rad}; \theta = 600 * t;$$



**Fig. 2.** Motion of the center of mass of the rotor over time.



**Fig.2 a.** Position of the first ball ( $\varphi_1$ ) over time.



**Fig. 2b.** Position of the second ball ( $\varphi_2$ ) over time

Analysis of the results obtained (Fig. 2a) shows that the rotor, which has a supercritical rotation velocity, over time makes steady motions in the vicinity of the amplitude value equal to  $A = 3 * 10^{-6} m$ . Along with this, the following can be noted: over time, the balancing balls occupy a certain position in the SBD tube (Figs. 2a, 2b), that is, one of the balls tends to position  $\varphi_1 = 0$ , and second ball occupies position  $\varphi_2 = -0.25 rad$ . Since the value of the oscillation amplitude of the center of mass of the rotor is quite small, the movement of the rotor can be considered as semi-balanced.

## 4 Conclusion

1. A mathematical model of a multi-row SBD was developed, considering not only the eccentricity of the center but also the case when the races with balancing balls have an axis of rotation.

2. The equations of motion were obtained in the form of the Lagrange equations in generalized coordinates. The conditions for stationary modes of motion of a multi-row SBD were also obtained. Some partial cases were considered and the location of balancing balls along the race was established; semi-balanced and balanced modes of rotor movement were considered.

4. At a small angular error of the SBD, the results obtained differ from the previous results by a term, which proportionally depends on the angular error of the balancing system.

5. In the case of rotor rotation with a supercritical angular velocity at a small angular error, when the SBD contains two balancing balls, a numerical result for specific values of the system parameters was obtained and an analysis of the results was performed.

## References

1. V.G. Bykov, A.S. Kovachev, Bulletin of St. Petersburg University. Ser.1. **1(4)** 579-588 (2014)
2. V.G. Bykov, Bulletin of St. Petersburg University. Ser.1. **2(4)** 90-101 (2006)
3. V.G. Bykov, A.S. Kovachev, Bulletin of St. Petersburg University. Ser. 1 **6(3)**, 452-461 (2019)

4. V.G. Bykov, Bulletin of St. Petersburg University. Ser. 1. Mathematics. Mechanics. Astronomy **3**, 89-96 (2010)
5. F.M. Dimentberg, K.T. Shatalov, A.A. Gusarov, *Machine vibrations* (Engineering., M., 1964)
6. A.S. Kelzon, L.M. Malinin, *Control of oscillations of rotors* (Polytechnic, St. Petersburg, 1992)
7. A.S. Kelzon, Yu.P. Tsimansky, V.I. Yakovlev, Dynamics of rotors in elastic supports, 280 (1982)
8. A.N. Gorbenko, Vibrations in engineering and technology **3(24)**, 33–36 (2002)
9. N.N. Zaitsev, D.N. Zaitsev, A.A. Makarov, Aerospace engineering **48**, 43-59 (2017)
10. V.V. Goncharov, G.B. Filimonikhin, Izvestia Vol. polytechnic university **326(12)**, 20-30 (2015)
11. V.P. Nesterenko, A.P. Sokolov, *Residual imbalance caused by the eccentricity of the race when the rotors are automatically balanced with balls. Dynamics of controlled mechanical systems* (OR, Irkutsk, 1983)
12. A.S. Kovachev, Bulletin of St. Petersburg State University. Series 1 **2**, 4 (2015)
13. G. Genta, *Dynamics of Rotating Systems* (Springer, 2005)
14. V. Pirogov, Eastern-European Journal of Enterprise Technologies **2-7(80)**, 49–63 (2016) DOI: 10.15587/1729
15. C.K. Sung, T.C. Chan, C.P. Chao, C.H. Lu, Mechanism and Machine Theory **69**, 115-126 (2013)
16. A.N. Gorbenko, Aviation and space technology and technics **8**, 71–76 (2011) DOI: [http://nbuv.gov.ua/UJRN/aktit\\_2011\\_8\\_16](http://nbuv.gov.ua/UJRN/aktit_2011_8_16)
17. M.E. Podolsky, S.V. Cherenkova, Theory of mechanisms and machines **12(1)**, 27–40 (2014)
18. J. Ehyaei, M.M. Moghaddam, Journal of Sound and Vibration **321**, 554–571 (2009) DOI: 10.1016/j.jsv.2008.10.019
19. D.J. Rodrigues, A.R. Champneys, M.I. Friswell, R.E. Wilson, International Journal of Non-Linear Mechanics **43**, 527–541 (2008) DOI: 10.1016/j.ijnonlinmec.2008.01.002
20. M.M. Mirsaidov, O.M. Dusmatov, M.U. Khodjabekov, *Mode Shapes of Hysteresis Type Elastic Dissipative Characteristic Plate Protected from Vibrations Proceedings of FORM 2022. Lecture Notes in Civil Engineering* (Springer, Cham, 2023) DOI: [https://doi.org/10.1007/978-3-031-10853-2\\_12](https://doi.org/10.1007/978-3-031-10853-2_12)
21. M.M. Mirsaidov, O.M. Dusmatov, M.U. Khodjabekov, PNRPU Mechanics Bulletin **3**, 51-59 (2022) DOI: 10.15593/perm.mech/2022.3.06
22. M.M. Mirsaidov, M.N. Sidikov, K.M. Turajonov, E3S Web of Conferences **365**, 04017 (2023) DOI: <https://doi.org/10.1051/e3sconf/202336504017>