Third order non-linear difference equation with neutral term

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Abstract. This paper aims to investigate the oscillatory characteristics of a neutral third order nonlinear difference equation. Utilizing the comparison principle, we get some new standards that guarantee that any solution to the neutral difference equation oscillates or approaches zero. Applications are then examined to show that the key theorems are valid.

1 Introduction

Consider the third order nonlinear difference equation with neutral terms

$$\Delta\big(\phi(m)(\Delta(\nu(m)\Delta y(m)))^{\beta}\big) + \zeta(m)x^{\gamma}(\rho(m)) = 0,\ m \geq m_0 \tag{1}$$

where $y(m) = x(m) \pm \omega(m)x^{\lambda}(v(m))$ and β, γ, λ are the ratio of odd positive integers. We assume throughout the article that

H1) $\{\phi(m)\}, \{\nu(m)\}, \{\zeta(m)\}, \{\omega(m)\}$ are positive real sequences;

H2) v and ρ are positive integers $\ni \rho(m) < m, v(m) < m$ and $\lim_{m\to\infty} \rho(m) = \infty$;

H3) $\lim_{m\to\infty} A(m, m_1) \to \infty$ where $A(m, m_1) = \sum_{s=m_1}^{m-1} \frac{1}{\phi^{\frac{1}{\beta}}(s)}$;

H4) $\sum_{s=m_1}^{\infty} \zeta(s) = \infty$ or

$$\sum_{s=m_1}^{\infty} \frac{1}{\nu(s)} < \infty \text{ and } \sum_{s=m_1}^{\infty} \frac{1}{\nu(s)} \sum_{u=s_1}^{s-1} \left[\frac{1}{\phi(u)} \sum_{v=u_1}^{u-1} \zeta(v)\right]^{\frac{1}{\beta}} = \infty;$$

H5) Furthermore, $v(m)$ is strictly increasing and $h(m) = v^{-1}(\rho(m)) < m$ with $\lim_{m\to\infty} h(m) = \infty$:

A nontrivial real sequence $\{x(m)\}$ that is determined for each $m_1 \geq m_0$ and satisfies equation (1.1) is referred to as a solution of (1.1). If a solution $\{x(m)\}$ is neither eventually positive nor eventually negative, it is referred to as oscillatory; otherwise, it is referred to as nonoscillatory.

Because it is used in several fields of engineering and natural science, the oscillation theory of functional differential equations has attracted a lot of interest recently. Difference equations are used to approximate differential equations. It aids in the advancement of digital

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machines. Differential equations are used in many different fields of science, engineering, and statistics ([1], [3], [4], [5], [11], [12]). Further references for the oscillatory and asymptotic behaviour of the second order difference equation include ([2], [6], [7], [8], [13], [14]) Numerous areas of science and mathematics, including population dynamics, delayed network system dynamics, and others, can benefit from the application of neutral delay difference equations. For neutral difference equations, various writers have recently investigated novel oscillation conditions [10]. By utilising the comparison theorem, our goal in this study is to give the oscillation and asymptotic results for equation (1.1).

2 Main Results

The demonstration of our main result depends heavily on the next lemma.

Lemma 2.1. (see [9]) "Let $\zeta(m)$ be a positive real sequence and let ρ be a positive integer. If the difference inequality

$$\Delta y(m) + \zeta(m) x^{\gamma}(\rho(m)) \leq 0,$$

has an eventually positive solution, then the difference equation

$$\Delta y(m) + \zeta(m) x^{\gamma}(\rho(m)) = 0,$$

has an eventually positive solution".

Theorem 2.2. Let $y(m) = x(m) + \omega(m) x^{\lambda}(v(m))$ and (H1) - (H4) hold. Suppose that $\lambda \leq 1, \lim_{m\to\infty} \omega(m) = 0$ and $\exists c \in (0,1)$ such that

$$\Delta U(m) + c^{\gamma} \zeta(m) \left(\sum_{s=m_5}^{m-1} \frac{A(s,m_4)}{\nu(s)} \right)^{\gamma} U^{\frac{\gamma}{\beta}}(\rho(m)) = 0 \tag{2}$$

is oscillatory. Then, every (1) solution is oscillatory or approaches to zero.

Proof. Let $\{x(m)\}$ is a non oscillatory solution of (1). We can posit that $x(m) > 0$, then there exists $m_1 \geq m_0$ such that $x(m) > 0, x(\rho(m)) > 0$ and $x(v(m)) > 0 \forall m \geq m_1$. From (1), we have

$$\Delta\big(\phi(m)(\Delta(\nu(m)\Delta y(m)))^{\beta}\big) = -\zeta(m) x^{\gamma}(\rho(m)) \leq 0\,, m \geq m_1 \tag{3}$$

which implies $\phi(m)\Delta(\nu(m)\Delta y(m))^{\beta}$ is nonincreasing and of one sign. We claim that, $\exists\, m_2 \geq m_1 \ni \Delta(\nu(m)\Delta y(m)) > 0$ for all $m \geq m_2$. Contrarily, suppose that there is $m_2 \geq m_1$ and a constant $e > 0 \ni$

$$\phi(m)(\Delta(\nu(m)\Delta y(m)))^{\beta} \leq -e, \quad m \geq m_2$$
$$(\Delta(\nu(m)\Delta y(m)))^{\beta} \leq -e \frac{1}{\phi(m)}, \quad m \geq m_2$$

Summing up the last inequality from m_2 to $m - 1$ and using (H3), thus

$$\nu(m)\Delta y(m) \leq \nu(m_2)\Delta y(m_2) - e \sum_{s=m_2}^{m-1} \frac{1}{\phi(s)^{\frac{1}{\beta}}} \to -\infty \text{ as } m \to \infty,$$

Again summing from m_3 to $m - 1$, we obtain

$$y(m) \leq y(m_3) - e\sum_{s=m_3}^{m-1}\frac{1}{\nu(s)}\sum_{u=s_3}^{s-1}\frac{1}{\phi(u)^{\frac{1}{\beta}}} \to -\infty \text{ as } m \to \infty,$$

which is a contradiction. Hence $\Delta(\nu(m)\Delta y(m)) > 0$ for all $m \geq m_2$.

Now, the following two cases will be analyzed

I $y(m) > 0, \Delta y(m) > 0$, or
II $y(m) > 0, \Delta y(m) < 0$.

Case I: In this case $y(m) = x(m) + \omega(m)x^{\lambda}(\nu(m))$, we carry

$$\begin{aligned} x(m) &= y(m) - \omega(m)x^{\lambda}(\nu(m)) = y(m)\left(1 - \frac{\omega(m)x^{\lambda}(\nu(m))}{y(m)}\right) \\ &\geq y(m)\left(1 - \frac{\omega(m)y^{\lambda}(m)}{y(m)}\right) = y(m)\left(1 - \frac{\omega(m)}{y^{1-\lambda}(m)}\right) \end{aligned}$$

In light of the fact that $\{y(m)\}$ is a nondecreasing and there exists a constant such that $y(t) \geq k$, where $k > 0$. Considering this, the last inequality becomes

$$x(m) \geq \left(1 - \frac{\omega(m)}{k^{1-\lambda}}\right)y(m)$$

Or

$$x(m) \geq cy(m),$$

where $c \in (0,1)$. Now, combining the inequalities (2) and (3), we have

$$\Delta\left(\phi(m)(\Delta(\nu(m)\Delta y(m)))^{\beta}\right) \leq -c^{\gamma}\zeta(m)y^{\gamma}(\rho(m)). \quad (4)$$

Hence $\Delta\left(\phi(m)(\Delta(\nu(m)\Delta y(m)))^{\beta}\right) \leq 0, \ \forall m \geq m_3$ and so, we have $\Delta(\nu(m)\Delta y(m)) > 0, \Delta y(m) > 0, \ \forall m \geq m_4$. It follows that

$$\begin{aligned} \nu(m)\Delta y(m) &= \nu(m_4)\Delta(m_4) + \sum_{s=m_4}^{m-1}\frac{\left(\phi(s)(\Delta(\nu(s)\Delta y(s)))^{\beta}\right)^{\frac{1}{\beta}}}{\phi^{\frac{1}{\beta}}(s)} \\ &\geq \phi^{\frac{1}{\beta}}(m)\Delta(\nu(m)\Delta y(m))\sum_{s=m_4}^{m-1}\frac{1}{\phi^{\frac{1}{\beta}}(s)}. \end{aligned}$$

Using the condition (H3)

$$\nu(m)\Delta y(m) \geq A(m, m_4)\phi^{\frac{1}{\beta}}(m)\Delta(\nu(m)\Delta y(m)). \quad (5)$$

Summing the previously mentioned inequality to m_5 to $m-1$, we get

$$\begin{aligned} y(m) &\geq \sum_{s=m_5}^{m-1}\frac{A(s,m_4)}{\nu(s)}\phi^{\frac{1}{\beta}}(s)\Delta(\nu(s)\Delta y(s)) \\ &\geq \sum_{s=m_5}^{m-1}\frac{A(s,m_4)}{\nu(s)}\left[\phi^{\frac{1}{\beta}}(m)\Delta(\nu(m)\Delta y(m))\right]. \end{aligned} \quad (6)$$

Combining (4) and (6), we obtain

$$\Delta U(m)+c^{\gamma}\zeta(m)\left(\sum_{s=m_5}^{m-1}\frac{A(s,m_4)}{v(s)}\right)^{\gamma}U^{\frac{\gamma}{\beta}}(\rho(m))\le 0,$$

where $U(m)=\phi(m)(\Delta(v(m)\Delta y(m)))^{\beta}$. Hence by Lemma 1 the corresponding equation

$$\Delta U(m)+c^{\gamma}\zeta(m)\left(\sum_{s=m_5}^{m-1}\frac{A(s,m_4)}{v(s)}\right)^{\gamma}U^{\frac{\gamma}{\beta}}(\rho(m))=0,$$

has a positive answer as well, which is the opposite of the hypothesis.

Case II: Since $\{y(m)\}$ is a positive sequence such that $\lim_{m\to\infty}y(m)=b\ge 0$. Now we have to prove that $b=0$. Assume that $b>0$. From equation (1)

$$\Delta\big(\phi(m)(\Delta(v(m)\Delta y(m)))^{\beta}\big)=-\zeta(m)x^{\gamma}(\rho(m))\le 0.$$

Summing the aforementioned inequality from m_1 to $m-1$, we carry

$$\phi(m)(\Delta(v(m)\Delta y(m)))^{\beta}\le -b^{\gamma}\sum_{s=m_1}^{m-1}\zeta(s),$$

$$\Delta(v(m)\Delta y(m))\le -b^{\frac{\gamma}{\beta}}\left(\frac{1}{\phi(m)}\sum_{s=m_1}^{m-1}\zeta(s)\right)^{\frac{1}{\beta}}.$$

Again Summing the last inequality, we have

$$(v(m)\Delta y(m))\le -b^{\frac{\gamma}{\beta}}\left(\sum_{s=m_1}^{m-1}\frac{1}{\phi(s)}\sum_{u=s_1}^{s-1}\zeta(u)\right)^{\frac{1}{\beta}}.$$

Again summing from m_1 to ∞, we obtain

$$y(m)\le -b^{\frac{\gamma}{\beta}}\sum_{s=m_1}^{\infty}\frac{1}{v(s)}\left[\sum_{u=s_1}^{s-1}\frac{1}{\phi(u)}\sum_{v=u_1}^{u-1}\zeta(v)\right]^{\frac{1}{\beta}}.$$

Using the condition (H4) in the aforementioned inequality, it becomes $y(m)\le -\infty$ which is a contradiction. Hence $\lim_{m\to\infty}y(m)=b=0$. The proof is completed.

Corollary 2.3. Let $\lim_{m\to\infty}\omega(m)=0, \gamma\le\beta$ and the condition (H1) to (H4) hold. If

$$\liminf_{m\to\infty}\sum_{u=\rho(m)}^{m-1}\zeta(u)\left(\sum_{s=m_0}^{\rho(u)-1}\frac{A(s,s_0)}{v(s)}\right)^{\gamma}=\infty, \qquad (7)$$

then (1.1) has oscillatory solution or tends to zero.

Theorem 2.4. Let $y(m)=x(m)-\omega(m)x^{\lambda}(v(m))$ here $\lambda\le 1$ and (H1) - (H5) hold. Moreover, suppose that there exist $\theta\in(0,1)$ and a positive sequence $\psi(m)\ni h(m)\le\psi(m)<m$ with $\lim_{m\to\infty}\psi(m)=\infty$ and there exists a constant $\theta\in(0,1)$. If

$$\Delta X(m)+\zeta(m)\left(\sum_{s=m_5}^{m-1}\frac{A(s,m_4)}{\nu(s)}\right)^{\gamma}X^{\frac{\gamma}{\beta}}(\rho(m))=0$$

And

$$\Delta W(m)\leq \theta^{\frac{\gamma}{\lambda}}\Gamma(m)\frac{h^{\frac{\gamma}{\lambda}}(m)}{\nu^{\frac{\gamma}{\lambda}}(m)}A(h(m),\psi(m))^{\frac{\gamma}{\lambda}}\left[W^{\frac{\gamma}{\lambda\beta}}(\psi(m))\right]=0,$$

Where

$$\Gamma(m)=\frac{\zeta(m)}{\omega^{\frac{\gamma}{\lambda}}(\nu^{-1}(\rho(m)))}$$

are oscillatory, then the solution of (1) is oscillatory or approaches to zero.

Proof. Let $\{x(m)\}$ is a nonoscillatory solution of (1). We may assume that $x(m)>0$, then there exists $m_1\geq m_0$ such that $x(m)>0, x(\rho(m))>0$ and $x(\nu(m))>0\forall m\geq m_1$. From (1.1), we have

$$\Delta\big(\phi(m)(\Delta(\nu(m)\Delta y(m)))^{\beta}\big)=-\zeta(m)x^{\gamma}(\rho(m))\leq 0,\ \ m\geq m_1 \tag{8}$$

since $\big(\phi(m)(\Delta(\nu(m)\Delta y(m)))^{\beta}\big)$ is nonincreasing and of one sign. From the Proof of Theorem 2.2, $\Delta(\nu(m)\Delta y(m))>0$ for all $m\geq m_2$. The next four cases will be examined now.

$$\begin{aligned}
\text{I } y(m) &> 0, \Delta y(m)>0,\\
\text{II } y(m) &> 0, \Delta y(m)<0,\\
\text{III } y(m) &< 0, \Delta y(m)>0,\\
\text{IV } y(m) &< 0, \Delta y(m)<0.
\end{aligned}$$

Case I: Here $y(m)=x(m)-\omega(m)x^{\lambda}(\nu(m))$. Since $x(m)\geq y(m)$. Thus

$$\Delta\big(\phi(m)(\Delta(\nu(m)\Delta y(m)))^{\beta}\big)=-\zeta(m)x^{\gamma}(\rho(m))\leq -\zeta(m)y^{\gamma}(\rho(m)) \tag{9}$$

Since $\Delta\big(\phi(m)(\Delta(\nu(m)\Delta y(m)))^{\beta}\big)\leq 0$ for all and so, we have $\Delta\big(\nu(m)\Delta y(m)\big)0$, $\Delta y(m)>0$ for all $m\geq m_4$. It follows that

$$\nu(m)\Delta y(m)=\nu(m_4)\Delta y(m_4)+\sum_{s=m_4}^{m-1}\frac{\big(\phi(s)(\Delta(\nu(s)\Delta y(s)))^{\beta}\big)^{\frac{1}{\beta}}}{\phi^{\frac{1}{\beta}}(s)}$$

$$\geq \phi^{\frac{1}{\beta}}(m)(\Delta(\nu(m)\Delta y(m)))\textstyle\sum_{s=m_4}^{m-1}\frac{1}{\phi^{\frac{1}{\beta}}(s)}.$$

Using the condition (H3)

$$\nu(m)\Delta y(m)\geq A(m,m_4)\phi^{\frac{1}{\beta}}(m)(\Delta(\nu(m)\Delta y(m))) \tag{10}$$

Summing the previously noted inequality to m_5 to $m-1$, we get

$$
\begin{aligned}
y(m) &\geq \sum_{s=m_5}^{m-1} \frac{A(s,m_4)}{\nu(s)} \phi^{\frac{1}{\beta}}(s)(\Delta(\nu(s)\Delta y(s))) \\
&\geq \sum_{s=m_5}^{m-1} \frac{A(s,m_4)}{\nu(s)} \left[\phi^{\frac{1}{\beta}}(m)(\Delta(\nu(m)\Delta y(m)))\right].
\end{aligned} \tag{11}
$$

Combining (9) and (11), we obtain

$$\Delta X(m) + \zeta(m)\left(\sum_{s=m_5}^{m-1} \frac{A(s,m_4)}{\nu(s)}\right)^{\gamma} X^{\frac{\gamma}{\beta}}(\rho(m)) \leq 0,$$

where $X(m) = \phi(m)\Delta(\nu(m)\Delta y(m))^{\beta}$. Hence by Lemma 1 the corresponding equation

$$\Delta X(m) + \zeta(m)\left(\sum_{s=m_5}^{m-1} \frac{A(s,m_4)}{\nu(s)}\right)^{\gamma} X^{\frac{\gamma}{\beta}}(\rho(m)) = 0,$$

has a positive as well, which is contradiction.

Case II: By Case II of Theorem 2.2, the solution of (1.1) tends to zero.

Case III: Assume that $z(m) = -y(m)$. Obviously, $\Delta z(m) = -\Delta y(m) < 0$ and $\Delta(\nu(m)\Delta z(m)) < 0$. This impossible due to $\Delta(\nu(m)\Delta z(m)) > 0$

Case IV: If we put $z(m) = -y(m)$, then

$$z(m) = -[x(m) - \omega(m)x^{\lambda}(\upsilon(m))] \leq \omega(m)x^{\lambda}(\upsilon(m))$$

or

$$x(\upsilon(m)) \geq \left(\frac{z(m)}{\omega(m)}\right)^{\frac{1}{\lambda}},$$

Hence

$$x(m) \geq \left(\frac{z(\upsilon^{-1}(m))}{\omega(\upsilon^{-1}(m))}\right)^{\frac{1}{\lambda}}.$$

Therefore equation (1) becomes,

$$
\begin{aligned}
\Delta\left(\phi(m)(\Delta(\nu(m)\Delta z(m)))^{\beta}\right) &= \zeta(m)x^{\gamma}(\rho(m)) \\
&\geq \frac{\zeta(m)}{\omega^{\frac{\gamma}{\lambda}}(\upsilon^{-1}(\rho(m)))} z^{\frac{\gamma}{\lambda}}(\upsilon^{-1}(\rho(m))) \\
&\geq \Gamma(m)z^{\frac{\gamma}{\lambda}}(h(m)).
\end{aligned} \tag{12}
$$

Now for $m_2 \leq p \leq q$. We see that there exists a constant $\theta \in (0,1)$ such that

$$z(h(m)) \geq \theta h(m)\Delta z(h(m)). \tag{13}$$

Substituting (13) in (12), we get

$$
\begin{gathered}
\Delta\left(\phi(m)(\Delta(\nu(m)\Delta z(m)))^{\beta}\right) \geq \Gamma(m)\theta^{\frac{\gamma}{\lambda}} h^{\frac{\gamma}{\lambda}}(m)\left[\frac{\nu(h(m))\Delta z(h(m))}{\nu(h(m))}\right]^{\frac{\gamma}{\lambda}}, \\
\Delta\left(\phi(m)(\Delta(V(m)))^{\beta}\right) \geq \Gamma(m)\theta^{\frac{\gamma}{\lambda}} h^{\frac{\gamma}{\lambda}(m)}\left[\frac{V(h(m))}{\nu(h(m))}\right]^{\frac{\gamma}{\lambda}},
\end{gathered} \tag{14}
$$

where $V(m) = \nu(m)\Delta z(m)$.

For $p \geq q \geq m_2$, we see that

$$V(p)-V(q) \quad =-\sum_{s=p}^{q-1} \phi^{\frac{1}{\beta}}(s)\phi^{\frac{1}{\beta}}(s)\Delta V(s),$$
$$\geq A(p,q)\left[-\phi^{\frac{1}{\beta}}(q)\Delta(\nu(q)\Delta z(q))\right].$$

Setting $p=h(m)$ and $q=\psi(m)$ above inequality becomes

$$\nu(h(m))\Delta z(h(m)) \geq A(h(m),\psi(m))\left[-\phi^{\frac{1}{\beta}}(\psi(m))(\Delta(\nu(\psi(m))\Delta z(\psi(m))))\right]. \quad (15)$$

Combining (2.14) and (2.15), we get

$$\Delta\big(\phi(m)(\Delta(\nu(m)\Delta z(m)))^{\beta}\big) \geq$$
$$\theta^{\frac{\gamma}{\lambda}}\Gamma(m)\frac{h^{\frac{\gamma}{\lambda}}(m)}{\nu^{\frac{\gamma}{\lambda}}(m)}A(h(m),\psi(m))^{\frac{\gamma}{\lambda}}\left[-\phi^{\frac{1}{\beta}}(\psi(m))(\Delta(\nu(\psi(m))\Delta z(\psi(m))))\right]^{\frac{\gamma}{\lambda}}.$$

Thus,

$$\Delta W(m) \leq -\theta^{\frac{\gamma}{\lambda}}\Gamma(m)\frac{h^{\frac{\gamma}{\lambda}}(m)}{\nu^{\frac{\gamma}{\lambda}}(m)}A(h(m),\psi(m))^{\frac{\gamma}{\lambda}}\left[W^{\frac{\gamma}{\lambda\beta}}(\psi(m))\right],$$

where $W(m)=-\phi(m)\Delta(\nu(m)\Delta z(m))^{\beta}$. Since the equivalent equation likewise has a positive solution according to Lemma 1, this is in contradiction. Hence the evidence.

Corollary 2.5. Let $\lambda \leq 1$ and (H1) to (H5) hold. Moreover, assume that the positive sequence $\psi(m)$ such that $h(m) \leq \psi(m) \leq m$ with $\lim_{m\to\infty}\psi(m)=\infty$ and there exists a constant

$\theta \in (0,1)$. If

$$\liminf_{m\to\infty}\sum_{u=\rho(m)}^{m-1}\zeta(u)\left(\sum_{s=m_0}^{\rho(u)}\frac{A(s,m_3)}{\nu(s)}\right)^{\gamma}=\infty$$

And

$$\liminf_{t\to\infty}\sum_{u=\psi(m)}^{m-1}\zeta(u)\Gamma(u)\frac{h^{\frac{\gamma}{\lambda}}(u)}{\nu^{\frac{\gamma}{\lambda}}(u)}A(h(u),\psi(u))^{\frac{\gamma}{\lambda}}=\infty,$$

then (1) has oscillatory solution or tends to zero.

3.3 Examples

Example 3.1. Take the nonlinear equation

$$\Delta\left(\frac{1}{m+2}\left(\Delta m^2\Delta\left(x(m)+\frac{1}{m+1}x^{\frac{1}{5}}\left(\frac{m^3}{2}\right)\right)\right)^4\right)+x^4\left(\frac{m^3}{2}\right), \; m \geq 1, \quad (16)$$

where $\phi(m)=\frac{1}{m+2}, \nu(m)=m^2, \; \omega(m)=\frac{1}{m+1}, \; \nu(m)=\rho(m)=\frac{m^3}{2}, \; \zeta(m)=1, \beta=\gamma=4$, and $\lambda=\frac{1}{5}$. So clearly it satisfies the conditions (H1) and (H2). Now we have to check these values satisfy the conditons (H3), (H4) and corollary 2.3. Therefore,

$$\lim_{m\to\infty}\omega(m)=\frac{1}{m+1}=0$$

$$A(m,1)=\sum_{s=1}^{m-1}\frac{1}{\phi^{\frac{1}{\beta}}(s)}=\sum_{s=1}^{m-1}\frac{1}{\left(\frac{1}{s+2}\right)^{\frac{1}{4}}}$$

$$\lim_{m\to\infty}A(m,1)=\sum_{s=1}^{\infty}\frac{1}{\left(\frac{1}{s+2}\right)^{\frac{1}{4}}}=\infty$$

and

$$\liminf_{m\to\infty}\sum_{u=\frac{m^3}{2}}^{m-1}\left(\sum_{s=1}^{\frac{u^3}{2}-1}(s^2)^{\frac{1}{4}}\right)=\infty$$

Hence, every solution of (3.1) is oscillatory or tends to zero because all the conditions and Corollary 2.3 are satisfied.

Example 3.2. Take the nonlinear equation

$$\Delta\left(\frac{1}{m+2}\left(\Delta m^2\Delta\left(x(m)-\frac{1}{m+1}x^{\frac{1}{5}}\left(\frac{m^3}{2}\right)\right)\right)^4\right)+x^4\left(\frac{m^3}{4}\right),\ m\geq 1 \qquad (17)$$

where $\phi(m)=\frac{1}{m+2}, v(m)=m^2, \omega(m)=\frac{1}{m+1}, v(m)=\frac{m^3}{2}, \rho(m)=\frac{m^3}{4}, \zeta(m)=1,\ \beta=\gamma=4$, and $\lambda=\frac{1}{5}$. So $h(m)=\frac{m^3}{2}$ and we take $\psi(m)=\frac{3m^3}{4}$. Now

$$\liminf_{m\to\infty}\sum_{u=\frac{m^3}{2}}^{m-1}\left(\sum_{s=1}^{\frac{u^3}{2}-1}(s^2)^{\frac{1}{4}}\right)=\infty$$

and

$$\liminf_{t\to\infty}\sum_{u=\frac{3m^2}{4}}^{m-1}\frac{1}{\left(\frac{1}{m+1^5}\right)\left(\frac{8}{m^3}\right)}\frac{(m^3)^{20}}{(2m^2)^{20}}\sum_{s=\frac{3m^3}{4}}^{\frac{m^3}{2}-1}\frac{1}{\frac{1}{(s+2)^4}}=\infty$$

Therefore every solution of (3.2) is oscillatory or tends to zero.

4 Conclusion

For the oscillatory behavior of solutions to a nonlinear equation with a neutral term, we have established some new criteria in this paper. The established findings are brand-new and add to earlier findings in the literature.

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