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An improved finite-difference scheme for the conservation equations of matter

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Abstract. The finite-difference scheme of directed differences (the Courant-Isaacson-Ries scheme), which is widely used in the practice of aerohydrodynamic calculations, is studied theoretically and on the example of test problems. We applied the commonly used in practice Courant-Isakson-Ries directional difference scheme that allowed us to find and show distributions of velocities where the laws of the matter conservation are violated in the calculations in solving the matter conservation equations or the correspondence of the obtained solutions to the most general practical understandings on the essence of the matter transfer. A scheme free from the shortcomings of the Courant-Isaacson-Ries scheme has been constructed, tested, and proposed for use in aerohydrodynamic calculations by the finite difference method. Moreover, all the valuable properties of this well-known scheme are preserved. Among the maintained properties: are transportability, conservatism, stability in calculations, invariance, and adequacy of the essence of the physical phenomenon of the transfer of matter in space. The disadvantages of the new finite-difference scheme proposed for solving the equations of conservation of matter should be considered: an increase in the required RAM for storing electronic means of calculating information about the velocity field in memory and an increase in the number of calculations needed.

1 Introduction

A generalization of Newton's second law to spatial motion and deformation during this motion of a spatially connected medium was made by Leonard Euler [1]. Analysis and generalization of the results of previous works show that the acceleration of any point of a connected medium is equal to the stress tensor divergence in the vicinity of this point divided by the density of the medium around the point under study.

Thanks to the results of research by Navier and Stokes [2] to justify the generalized state of a rheological body called Newton's viscous fluid at the beginning of the 19th century (approximately 1810–1820), the equations of fluid motion appeared. These equations are called the "Navier-Stokes Equations." A new science was born - HYDROMECHANICS.

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For nearly 100 years, mathematicians have spent enormous efforts investigating and solving the Navier-Stokes equations. There was no question of solving the complete system of equations. Separate private, highly specialized tasks were solved. For example, the motion of a sphere in an infinite resting fluid or the slow movement of a viscous liquid (laminar).

But in many cases, the authors discarded the so-called inertial terms of the Navier-Stokes equations. Their nonlinearity was an insurmountable obstacle to obtaining analytical solutions. The following 100 years can be conditionally divided into two equal parts. In the first half of the twentieth century, scientific and technological progress required knowledge about gas and fluid movement. Aeronautics needed laws to calculate the flow of gas around various surfaces. A semi-experimental, semi-heuristic branch of hydromechanics, named HYDRAULICS, developed rapidly. It made it possible to calculate a broader range of phenomena than scientists could afford within the framework of classical fluid mechanics.

But classical hydromechanics also did not standstill. Simultaneously, with the advent of the first calculating devices (tube and even mechanical), a branch of hydromechanics began to form with the name of COMPUTATIONAL HYDROMECHANICS. It is the most interesting for us, and therefore we will dwell on its development in more detail. This science made it possible to make weather forecasts and calculate the transfer of climatic characteristics along the planet's surface along with the winds. The quality of weather forecasts has improved dramatically.

This happened due to Courant, Isaacson, Rees, Levy [3]. They developed and researched a finite-difference scheme that makes it possible to calculate the effect of both inertial terms in the equations of motion and convection terms in the equations of conservation of the mass of a moving medium.

The scheme revealed some shortcomings associated with the order of accuracy in the approximation of derivatives [4]. More than a dozen similar schemes have been developed, mainly pursuing the goal of increasing the level of approximation by finite-differences of partial derivatives, such as Schemes: Lax and Wendroff [5], Lax-Wendroff [6,7], leapfrog, classics, etc. [8]. But due to the simplicity of the algorithmization, the Courant-Isaacson-Rees scheme [9] has become dominant.

It has come to be used so frequently that it has been referred to as either a directed difference scheme or an “upstream” scheme for ease of reference. The mention of this scheme can be found in most problems devoted to calculating the circulation of the atmosphere and ocean [7,9,10]. Thanks to the research of Novosibirsk scientists under the leadership of G.I. Marchuk [10–12] and the achievements of mathematicians-computers Lodyzhenskaya, Godunov, Ryabenky, Richtmyer [4,13–15], the era of accelerated development of computational fluid mechanics began.

Weather forecasts require more and more powerful computers. The requirements of computational fluid mechanics were among the factors that contributed to the influx of investment and brains into the development of computer technology. Currently, there are about a dozen mathematical models of weather forecasting. In their mathematical description, one can find a mention about the usage of “upstream” schemes. Where there are no such mentions, in the results can be notice the behavior of these schemas if one is familiar with the operation of these schemas.

What can be found in these well-known and frequently used calculation schemes? This article is devoted to the answer to this question. It turns out that with a special and even frequently encountered distribution of velocities, the “upstream” scheme loses its conservatism when calculating conservative phenomena of momentum or substance transfer. This manifestation of non-conservatism in calculations can cast doubt on any result obtained because the law of conservation of matter is the basic fundamental law of our being. How so? How can one not notice such a problem that calls into question any calculation?

This article describes how this fundamental problem was hidden and masked in the calculations but was not discovered. Moreover, the paper presents a modification of the "upstream" scheme in an expanded non-divergent form free from the loss of conservatism. The proposed scheme retains all the advantages of the Courant-Isaacson-Rees scheme and does not contain the disadvantage of a possible loss of conservatism. The modified schema can increase the performance of predictive models. The scheme is a new calculation tool in computational fluid mechanics.

2 Materials and methods

This study used a standard method for solving the matter transfer equations (sometimes called the matter conservation equations) using finite difference approximations. Using this method allowed us to find velocity distributions leading to a loss of conservativeness in calculations for the scheme of directional differences most widely used in practical calculations. Based on the results obtained, a possible reason is given why the fatal problem of loss of conservatism has been hidden so far and continues to exist imperceptibly in most software products that calculate hydrodynamic processes.

The conducted studies made it possible to theoretically and practically show the reasons for the loss of conservatism in the calculations. In this case, the phenomenon of "sticking" the solution to zones of zero velocities was noted.

During the research, the method of mathematical analysis of differential equations and their finite-difference analogs, test calculations using the most straightforward tools for calculations, and comparing the calculation results with analytical solutions and general ideas about the essence of conservation laws during the transfer of matter were used.

Based on the results of the studies, modifications of calculation schemes are proposed that remain conservative for any velocity distributions. Their implementation in practice requires minor changes in the algorithms of the models. The result will be the possibility of increasing the computational time step due to the increased stability of the solution by reducing the factors perturbing the solution and, as a result, increasing the accuracy and efficiency of calculations.

Figures and tables, as originals of good quality and well contrasted, are to be in their final form, ready for reproduction, pasted in the appropriate place in the text. Try to ensure that the size of the text in your figures is approximately the same size as the main text (10 point). Try to ensure that lines are no thinner than 0.25 point.

3 Results and discussion

The equation of matter transfer in space is an integral part of the vast majority of mathematical models of aero hydrodynamics. These are the problems of weather forecasting [5,16,17], calculation of water exchange in the seas [18,19] and oceans [6,14,20], transport and distribution of pollutants in the atmosphere [21], and surface water bodies [4]. It is the most challenging part of the equations of motion and transfer equations for calculations. Let us consider the transport equation in more detail, discarding all other factors.

For the one-dimensional spatial case, the equation for the transport of matter S is as follows:

The first form of the mass conservation equation

$$\frac{\partial S}{\partial t} = -\frac{\partial (SV)}{\partial x}, \qquad (1a)$$

The second form (expanded form and identical to the first form)

$$\frac{\partial S}{\partial t} = -\left(V\frac{\partial S}{\partial x} + S\frac{\partial V}{\partial x}\right), \qquad (1b), \qquad (1)$$

where ∂ is a partial derivative,
t- time, sec,
x- spatial coordinate, m,
S- some mass of matter, kg,
V-velocity of matter, m/sec.

Let us show that the finite-difference analogs of the convective parts of Eq. (1), which are widely used in world practice, can lose their conservatism for some velocity distributions. Here we do not consider diffusion, which is often used to improve stability in calculations [22,23]. Natural diffusion is not considered, and computational diffusion is minimized.

We try to show that the most straightforward problems of aero hydrodynamics already in two-dimensional space almost always guarantee the formation of velocity fields, where the well-known schemes lose their conservatism.

Note that equation (1) is a record of the law of conservation of matter in differential form. This means that always and at any movement speed of a substance, its amount should not change. This mandatory property is called the property of conservatism [4,18].

First of all, let us write down the finite-difference analogs of Eq. (1), which are often used in practice and written in explicit form.

Explicit schemes are schemes in which the future distribution of matter is completely calculated based on the existing distribution of matter and known velocities of movement [3,6,19]. The most common calculation schemes are directional difference schemes [5,6,8,18,20].

The best-known finite-difference scheme of directed differences is called the "Courant-Isaacson-Rees scheme"[1,24]. This scheme is considered conservative, transportable, and stable under the Courant-Levy condition [3,8,13,23,25–27]. Transport schemes are schemes that, when applied, provide an imitation of the movement of numerical values characterizing the amount of a substance in the direction of the speed of movement [15,18]. Let us add one more important property to the analysis of the circuits - invariance. Invariance is the independence of the physical process and the calculation imitating it from the choice of coordinates.

For the equation(1a) of writing equation (1), we have finite-difference analog

$$S_{t+1,i} = S_{t,i} - \frac{\triangle t}{\triangle x} * \left(V_{t,i} * S_{t,i} - V_{t,i-1} * S_{t,i-1}\right), \quad if\ V_{t,i} > 0$$
$$S_{t+1,i} = S_{t,i} - \frac{\triangle t}{\triangle x} * \left(V_{t,i+1} * S_{t,i+1} - V_{t,i} * S_{t,i}\right), \quad if\ V_{t,i} < 0 \qquad (2)$$

For the equation (1b) of writing equation (1), we have finite-difference analog

$$S_{t+1,i} = S_{t,i} - \frac{\triangle t}{\triangle x} * \left(V_{t,i} * \left(S_{t,i} - S_{t,i-1}\right) + S_{t,i} * \left(V_{t,i+1} - V_{t,i}\right)\right), \quad ifV_{t,i} > 0$$
$$S_{t+1,i} = S_{t,i} - \frac{\triangle t}{\triangle x} * \left(V_{t,i} * \left(S_{t,i+1} - S_{t,i}\right) + S_{t,i} * \left(V_{t,i} - V_{t,i-1}\right)\right), if\ V_{t,i} < 0$$
(3)

Sometimes, researchers write one of the two inequalities for schemes (2) and (3) in a non-strict form, replacing ">" (or "<") with "≥" (or "≤"). Because in the strict forms in the inequalities “>” (or “<”) the velocity equal zero excluded from computations. Thus, an attempt is made to ensure the entire possible spectrum of velocities calculate fully. Let us show that this action either does not affect anything or can lead to the loss of the invariance of the calculation. The latter is also an unacceptable phenomenon in the calculations.

The equation (1a) is very often used in aerohydrodynamic models [6,11,12,14–18,22,25].

Note, that the equation (1b) and its finite-difference analog (from now on, the f.-d. analog) are rarely used due to the need for twice the number of calculations compared to the first form of the substance transfer equation with identical results in the case of similar distributions of velocities and invariance of the sign of the velocity. But the reasons for this ignorance, in our opinion, are much more serious.

But it is the second form of writing equation (1) that made it possible to propose a new conservative f.-d. analog for solving equation (1).

The conservatism of f.-d. analogs for transport equations are verified as follows. The calculated values of the substance at the future moment are written out for the sequence of calculation points, and their mutual destruction is checked during summation. Let us consider the distribution of velocities along the X-axis, at which the conservatism of finite-difference analogs (2) and (3) is violated. Suppose that for nodes with a serial number less than 7, the velocity of the substance is positive, and for nodes with a serial number of 7 and above, the velocity of the substance is negative.

Then f.-d. analog (2) for a group of nodes near node number 7 will look like this.

$$
\begin{aligned}
S_{t+1,4} &= S_{t,4} - \frac{\triangle t}{\triangle x} \cdot \left(V_{t,4} \cdot S_{t,4} - V_{t,3} \cdot S_{t,3}\right) \\
S_{t+1,5} &= S_{t,5} - \frac{\triangle t}{\triangle x} \cdot \left(V_{t,5} \cdot S_{t,5} - V_{t,4} \cdot S_{t,4}\right) \\
S_{t+1,6} &= S_{t,6} - \frac{\triangle t}{\triangle x} \cdot \left(V_{t,6} \cdot S_{t,6} - V_{t,5} \cdot S_{t,5}\right) \\
S_{t+1,7} &= S_{t,7} - \frac{\triangle t}{\triangle x} \cdot \left(V_{t,8} \cdot S_{t,8} - V_{t,7} \cdot S_{t,7}\right) \\
S_{t+1,8} &= S_{t,8} - \frac{\triangle t}{\triangle x} \cdot \left(V_{t,9} \cdot S_{t,9} - V_{t,8} \cdot S_{t,8}\right) \\
S_{t+1,9} &= S_{t,9} - \frac{\triangle t}{\triangle x} \cdot \left(V_{t,10} \cdot S_{t,10} - V_{t,9} \cdot S_{t,9}\right)
\end{aligned} \tag{4}
$$

Let's ignore the lonely terms "$V_{t,10} \cdot S_{t,10}$" and "$V_{t,3} \cdot S_{t,3}$". These terms must either interact with the boundary conditions or cancel with the terms appearing in the calculation of "$S_{t,3}$ " and "$S_{t,10}$ ".

But note that the terms "$V_{t,6}\, S_{t,6}$" and "$V_{t,8}\, S_{t,8}$" for a given change in the sign of the velocities at the point with index 7 is not reduced. This means that a scheme recognized as conservative and often used in practice under certain conditions loses its conservatism.

Let us write the formulas for calculating the distribution of matter “S”, provided that the second form of writing the equation for the transfer of matter (1) was used.

$$
\begin{aligned}
S_{t+1,4} &= S_{t,4} - \frac{\triangle t}{\triangle x} \cdot \left(V_{t,4} \cdot S_{t,4} - V_{t,4} \cdot S_{t,3} + V_{t,5} \cdot S_{t,4} - V_{t,4} \cdot S_{t,4}\right) \\
S_{t+1,5} &= S_{t,5} - \frac{\triangle t}{\triangle x} \cdot \left(V_{t,5} \cdot S_{t,5} - V_{t,5} \cdot S_{t,4} + V_{t,6} \cdot S_{t,5} - V_{t,5} \cdot S_{t,5}\right) \\
S_{t+1,6} &= S_{t,6} - \frac{\triangle t}{\triangle x} \cdot \left(V_{t,6} \cdot S_{t,6} - V_{t,6} \cdot S_{t,5} + V_{t,7} \cdot S_{t,6} - V_{t,6} \cdot S_{t,6}\right) \\
S_{t+1,7} &= S_{t,7} - \frac{\triangle t}{\triangle x} \cdot \left(V_{t,7} \cdot S_{t,8} - V_{t,7} \cdot S_{t,7} + V_{t,7} \cdot S_{t,7} - V_{t,6} \cdot S_{t,7}\right) \\
S_{t+1,8} &= S_{t,8} - \frac{\triangle t}{\triangle x} \cdot \left(V_{t,8} \cdot S_{t,9} - V_{t,8} \cdot S_{t,8} + V_{t,8} \cdot S_{t,8} - V_{t,7} \cdot S_{t,8}\right) \\
S_{t+1,9} &= S_{t,9} - \frac{\triangle t}{\triangle x} \cdot \left(V_{t,9} \cdot S_{t,10} - V_{t,9} \cdot S_{t,9} + V_{t,9} \cdot S_{t,9} - V_{t,8} \cdot S_{t,9}\right)
\end{aligned} \tag{5}
$$

As in the case of the scheme (4), we ignore the terms “$V_{t,4} \cdot S_{t,3}$” and “$V_{t,9} \cdot S_{t,10}$”. These terms will either have to interact with the boundary conditions or be destroyed when calculating the substance at points with indices “3” and “10”.

It can be seen that the terms “$V_{t,6} \cdot S_{t,5}$” and “$V_{t,6} \cdot S_{t,7}$” do not cancel if the velocity changes sign at point "7" and as written above.

Even this scheme, which is rare, but sometimes used in calculations, also loses its conservatism under certain conditions. Here it is considered only because it made it possible to create a new conservative scheme for calculating the motion of an admixture. But this will be discussed later in the text.

The one-dimensional calculation section is divided into parts so that there are 100 calculation nodes. Let the distance between nodes be one meter. In the first to the node section, the velocity is positive and equal one meter per second. In the section from the 51st node to the 100th node, the velocity is negative and equals minus one meter per second. On nodes with numbers “20”, “21”, “22,” the speed is reduced to +0.5m/s. On nodes with numbers "79", "80", "81" the speed increases to -0.5m/sec.

Decreasing the velocity in two zones of the calculated segment was done to see the behavior of finite difference schemes when calculating the passage of matter through these zones. In nodes with numbers “1,” “2,” “99,” “100,” there is no velocity of the substance to emphasize once again the impossibility of the substance entering the calculation zone from the outside. Although this protection, as it is easy to check, is redundant. The distribution of velocities is set to be symmetrical with respect to the center of the computational zone. If the distribution of matter at the initial moment is also symmetrical, then the solution must be symmetrical at any time. So, it will be possible to verify the invariance of the design schemes without additional calculations.

Figure 1 shows the distribution of velocity for each of 100 calculated nodes.

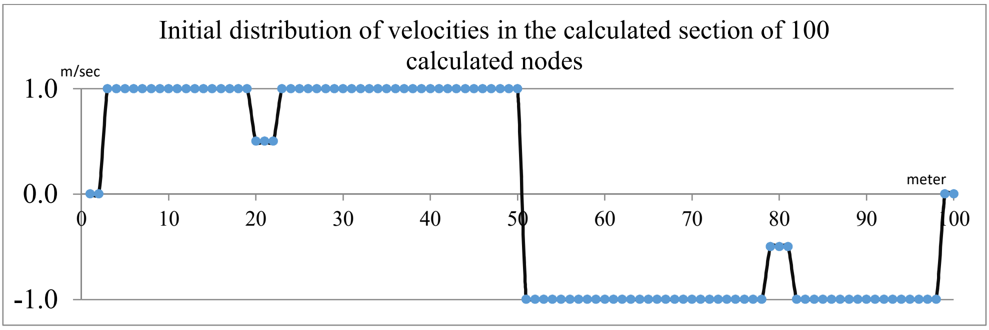

Fig. 1. Distribution of matter transfer rates for each of 100 calculated nodes.

For all 100 nodes, except for numbers “10” and “91”, the initial content of matter is set equal to zero. At points with numbers “10” and “91”, the initial content of the substance is set equal to “10”. This is a symmetrical distribution of matter relative to the center of the calculated area.

The solution must be transportable, conservative, symmetrical, and adequate to reality. Symmetry should be observed in the solution as a consequence of the symmetry in the velocity distribution and the initial distribution of matter. A physically justified solution will be considered if there are no oscillations in the solution. The fact is that sometimes conservatism in the amount of a substance can be combined with the appearance of unreasonably high values of the substance or even its negative values. It will be possible to see the manifestation of inadequacy for one of the schemes below.

We will use a time step equal to the maximum limit value of the time step according to the Courant-Levy criterion. That is, $\Delta t = \frac{\Delta x}{V_{max}} = 1.0 sec$.

It is clear that the substance, moving towards the center of the calculated segment from nodes with serial numbers “10” and “91”, will reach the center of the segment a little later than the 40thtime interval (less due to the fact that there is a section where the substance will move at a speed of 0.5 m/s instead of speed 1.0 meter/s).

Figure 2 shows the distribution of matter at the initial moment of time, for the 12th time interval (in zones of low speeds).

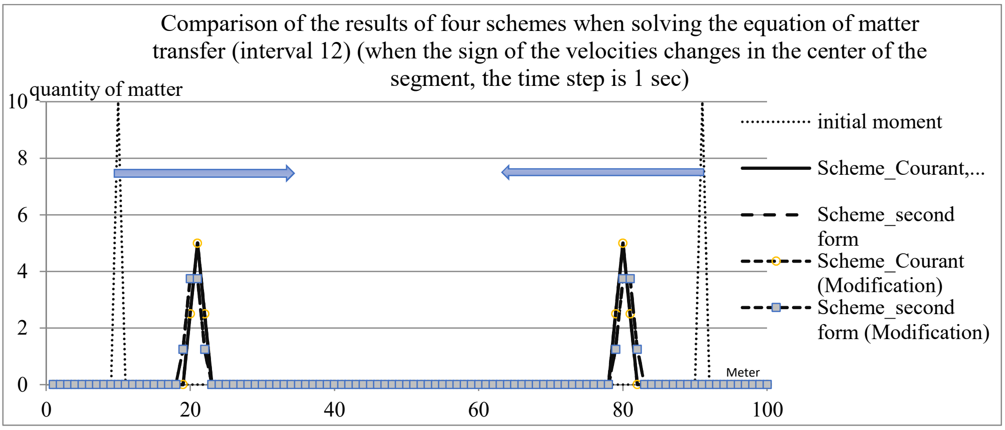

Fig. 2. Distribution of matter at the initial moment of time, and for the 12th time interval (in the zone of low speeds).

Figure 2 shows the symmetrical advance of matter towards the center of the computational zone. We notice some discrepancy in the results between the Courant schemes (with and without modification) and the schemes for the equation (1b) (with and without modification). It can be concluded that the scheme viscosity for the Courant scheme and the scheme for the equation (1a) does not manifest themselves in the same way in time but in the same way in action. Figure 3 shows the distribution of matter for the 30th time interval. All four circuits give identical results, which proves the sameness in the manifestation of circuit viscosity.

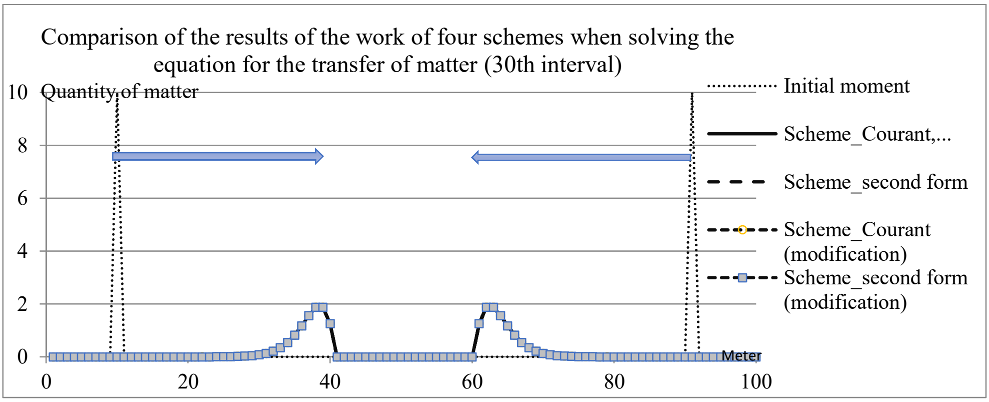

Fig. 3. Distribution of matter at the initial moment of time, and for the 30th time interval (after the zone of low speeds).

On the 41st time interval, the matter, as expected, reaches points with numbers "50" and "51", in which the velocities have different signs. On the 41st time interval, the conservativeness of the calculation is lost. The loss of conservatism at interval 41 is shown in figure 4.

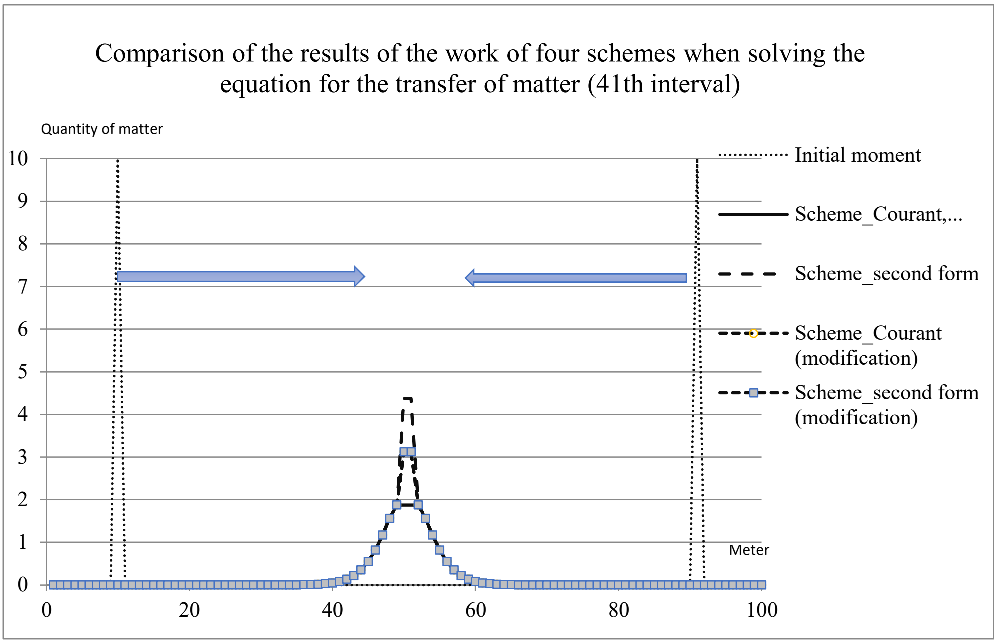

Fig. 4. Distribution of matter at the initial moment of time, for 41-time intervals.

The Courant-Isaacson-Rees scheme changes the substance contained in the calculated zone from 20 to 17.50, a decrease of mass (time step 1, space step 1, the velocity distribution is given in figure 1). The second form of writing equation (1) changes the substance contained in the calculation zone from 20 to 22.50, an increase of mass. And only after modification do both of these schemes retain the conservatism of the calculation. When applying the modified schemes, the substance, as it was in the initial distribution of 20 units, remained the same after transferring the substance to the points where the velocities change sign. The amount of substance for all schemes is positive for all time intervals and the entire computational zone. That is, all results are adequate. Strict or non-strict inequality was used in schemes (2) and (3) and this does not matter since the velocities in the zone of loss of conservatism were always different from zero.

It also becomes clear why hydrodynamicists avoid using the second form of the matter conservation equation (1) and mainly use only the first form. The disappearance of matter cannot crash the computational process. An uncontrolled increase in the amount of a substance can cause an emergency termination of calculations. For the first form of writing Eq. (1), there is even a very optimistic name, the "divergent" form of writing the conservation equations [13]. There is even a practice of switching from the usual forms of writing equations to the "divergent" form only to ensure the possibility of using the Courant-Isaacson-Rees scheme [18,24].

The Courant-Isaacson-Rees scheme and the scheme for the second form of the conservation equation (1) after modifications show transportability, conservatism, stability, invariance (symmetry), adequacy.

Equations (7), and (8) reveal the modification and show the structures of the new schemes,

$$\begin{aligned} U_{t,i} &= V_{t,i} \; if \; V_{t,i} \geq 0, \\ U_{t,i} &= 0 \; if \; V_{t,i} > 0, \\ W_{t,i} &= V_{t,i} \; if \; V_{t,i} \leq 0, \\ W_{t,i} &= 0 \; if \; V_{t,i} > 0; \end{aligned} \qquad (6)$$

For the Courant-Isaacson-Rees scheme

$$S_{t+1,i} = S_{t,i} - \frac{\triangle t}{\triangle x} \cdot \begin{bmatrix} U_{t,i} \, S_{t,i} - U_{t,i-1} \, S_{t,i-1} \\ + \\ W_{t,i+1} \, S_{t,i+1} - W_{t,i} \, S_{t,i} \end{bmatrix} \qquad (7)$$

And for the scheme of the second form of writing the conservation equation (1)

$$S_{t+1,i} = S_{t,i} - \frac{\triangle t}{\triangle x} \cdot \begin{bmatrix} U_{t,i} * \left(S_{t,i} - S_{t,i-1}\right) + S_{t,i} * \left(U_{t,i+1} - U_{t,i}\right) \\ + \\ W_{t,i} * \left(S_{t,i+1} - S_{t,i}\right) + S_{t,i} * \left(W_{t,i} - W_{t,i-1}\right) \end{bmatrix} \qquad (8)$$

Where $U_{t,i}$, $W_{t,i}$ are intermediate variables calculated through the value of the velocity of the matter and depending on the sign of this velocity.

The following objection may arise. Velocities having differing signs in neighboring nodes are rare in practice. This is an entirely wrong objection. In any even two-dimensional problem, where there is an expansion or contraction of a moving flow, the velocity components perpendicular to the main motion will have neighboring points at which the sign of the velocity's changes to the opposite. Only in a pipe of constant diameter or a channel of a continuous cross-section will there be no situations leading to a loss of conservatism in the calculations when using directed finite difference schemes. The common of which is the Courant-Isaacson-Rees scheme [18,24]. We can conclude that schemes (7) and (8) are suitable for calculations in which there are equations for the transfer and conservation of matter. However, it is not. Consider the following velocity diagram. Let zero velocities be set in the central symmetrical section of the calculation area (points with numbers "50" and "51"). In practice, this means either an island in the middle of a river or a solid body at rest in a moving stream of gas or liquid. The velocity distribution diagram is shown in figure 5.

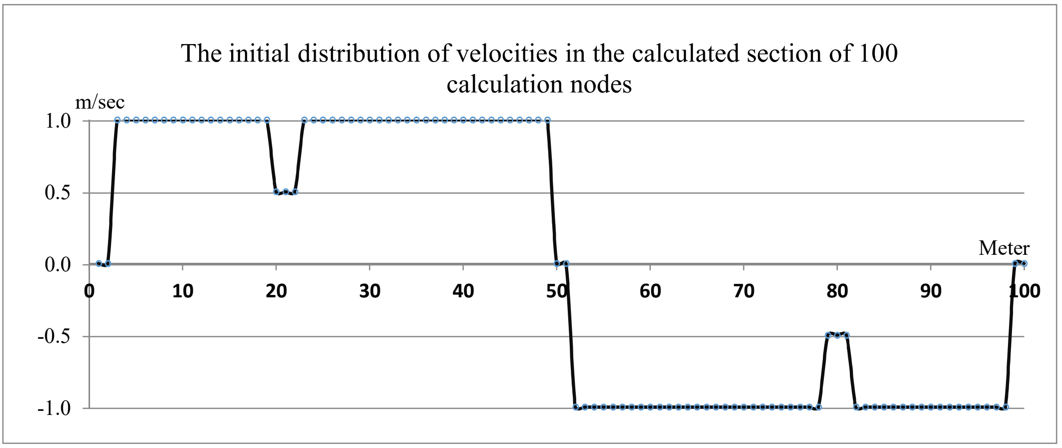

Fig. 5. Velocity diagram with a zone of zero velocities in the center of the computational zone.

First of all, we note the loss of invariance (symmetry in this case) for the Courant-Isaacson-Rees scheme on 41 calculation intervals. The loss of invariance is shown in figure 6.

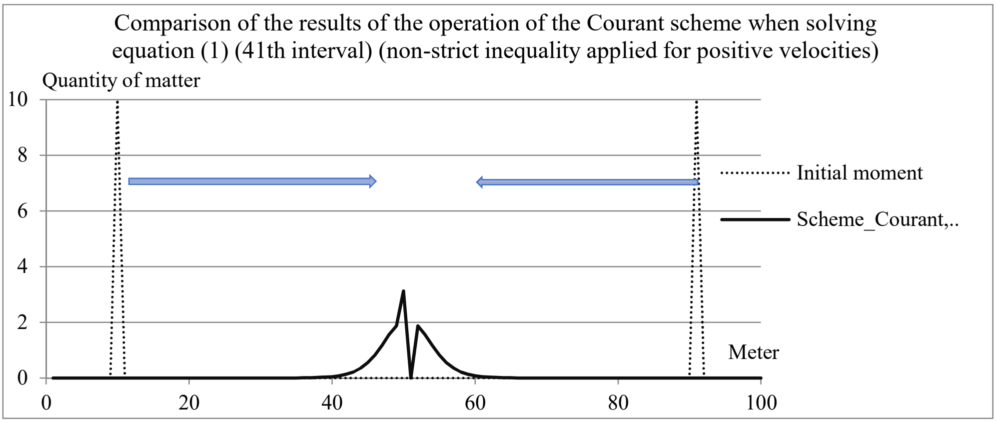

Fig. 6. Loss of invariance of the Courant-Isaacson-Rees scheme when approaching the zone of zero velocities.

We note an interesting fact. For positive velocities, conservatism is preserved when approaching the section of zero velocities. The left peak is 10, the right one is 6.88.

It is impossible to maintain conservatism for both positive velocities and negative velocities at the same time. Even if we write a non-strict inequality for both parts of the scheme (2), they are calculated sequentially. Therefore, the last calculation will suppress the previous calculations, and the loss of conservatism will not disappear. Let's introduce a correction into the test problem in which there is a section with zero velocities. Let the velocities change their sign to the opposite on the 43rd time interval. We will only check the operation of the modified schemes since neither the Courant-Isaacson-Rees scheme nor the scheme of the second form of the matter conservation equation (3) passed the first test.

In this test problem, after the 43rd interval, we expect the substance to leave the area of zero velocities and the movement of these peaks to the edges of the computational zone. Figure 7 shows the distribution of matter in the 50th time interval (50 seconds).

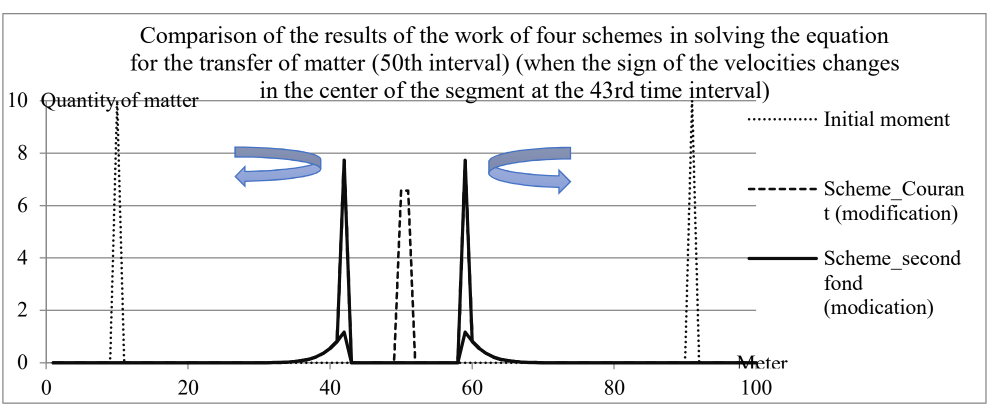

Fig. 7. Substance distribution on the 50th time interval (50 seconds) for modifications of schemes (2) and (3) made according to the rule (6), (7), (8).

Despite the conservatism of both modifications of schemes (6), (7), (8), the inadequacy of the calculation is also visible. In the case of modification (6), (7), the substance cannot leave the area of zero velocities and remains there forever. Only the substance that has not yet reached this “dead zone” of zero velocities can be transported by the scheme (6), (7) to the edges of the calculated area.

The modification in Eqs.(6)and(8), carried out for the scheme of directed differences of the second form of the equation of conservation of matter, passed both test checks without remarks. In this way, it surpasses all other schemes studied, and it is this scheme that is recommended for use in problems of aerohydrodynamics.

The loss of conservatism when solving inherently conservative equations is a very serious problem. The question may arise - how has such a problem not been studied so far, and how did it manage to be hidden from researchers?

The fact is that the smaller the absolute values of the velocities in the sign change zone, the smaller the loss of conservatism. This conclusion can be drawn by estimating the terms that cannot mutually annihilate in the velocity sign change zone (4.5). Diffusion processes blur the zones of loss of conservatism, disguising them as rounding errors in calculations. Computational artificial viscosity is also involved in making the areas of loss of conservatism.

4 Conclusions

Applying the calculation schemes developed in this article will make the aerohydrodynamic calculations more stable. Fluctuations in the solution will not need to be smeared and smoothed either by artificial diffusion or by computational viscosity. The time step can be increased, which will naturally increase the efficiency of aerohydrodynamic calculations. The smaller the absolute values of the velocities that change their sign at neighboring points, the smaller the loss of conservatism. Transverse velocities are usually always much less than the longitudinal and central motions of the flow of matter. This circumstance usually also masks the loss of conservatism. The discovered non-conservatism is traditionally explained by rounding errors in figures when they are calculated on a computer.

But it doesn't turn out that way. The loss of conservatism is conceptually embedded in the directed finite-difference analogs of the convective terms of the equations of conservation of matter. The new scheme (6) is free from this shortcoming.

The result of applying the new scheme will be using time steps as close as possible to the limiting time step determined by the Courant-Levy criterion. This means that a natural result of applying the new scheme will be an increase in the efficiency of solving aerodynamic problems. Any of the modern two-dimensional and three-dimensional computational aerohydrodynamic systems that use the scheme of directed differences in calculations after minor code changes will increase the accuracy and efficiency of calculations. Applying the newly proposed modification of the directional difference scheme will require minimal code changes in aerodynamic models using the Courant-Isaacson-Rees scheme.

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