CONTINUED FRACTIONS OF HIGHER ORDER POLYGONAL NUMBERS WITH RESPECT TO ORDER AND RANK

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Abstract: Developing number sequence based on polygonal numbers is an enthusiastic field in number theory. As tetrahedral numbers are similar to pyramids, one of the Seven wonders of the World, yields a unique copiousness in its suitability. In number theory study of pyramidal numbers vary in richness and variety. Also the study of continued fractions is a fastly developing field.

Keywords: *continued fraction, pyramidal number, order, rank.*

Notations

1. $\langle p_0, p_1, p_2, p_3, \ldots., p_n \rangle -$ continued fraction expansion
2. $PN(n, r)$ $-$ pyramidal number of order $'n'$ and rank $'r'$
3. $PN(3, r)$ $-$ triangular pyramidal number
4. $PN(4, r)$ $-$ square pyramidal number

1. Introduction

Representation of rational numbers as continued fractions is continuously studied by various mathematicians. The convergence of continued fractions is specially studied and applied in finding the best rational gear ratio approximations. Higher dimensional polygonal numbers namely pyramidal numbers combined with continued fractions is really mind boggling. Finding solution of cubic equations with figurate numbers as coefficients in terms of continued fractions is found in. Centered polygonal numbers taken in various forms as continued fraction as given in [3,9] is really a motivating factor for this paper.

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2. Pyramidal number

The polygonal number of higher order namely, pyramidal number of order n and rank r is represented by the formula

$$PN(n,r)=\frac{n(n+1)\big(n(r-2)-(r-5)\big)}{6}$$

2.1 Continued Fraction

An expression of the form

$$\frac{a}{b}=p_0+\cfrac{q_0}{p_1+\cfrac{q_1}{p_2+\cfrac{q_2}{p_3+\cfrac{q_3}{\ddots}}}}$$

where p_i, q_i are real or complex numbers is called a continued fraction.

2.1 Theorem

If $'n'$ and $'r'$ are order and rank of the corresponding pyramidal numbers then the continued fraction of $\frac{PN(n,r)}{PN(n+1,r)}$ is given by

$$\frac{PN(n,r)}{PN(n+1,r)}=\begin{cases}\langle 0,1,2k-1,\frac{(3k-1)r-3(2k-1)}{k}\rangle, if\ n=6k-2\\ \langle 0,1,2k-1,1,\frac{(6k-1)r-(6k-5)}{2r(6k-1)-\big(4(6k-1)+3(2k-1)\big)}\rangle, if\ n=6k-1\\ \langle 0,1,2k-1,1,\frac{2kr-(3k-1)}{k(r-3)}\rangle, if\ n=6k\\ \langle 0,1,2k,\frac{(6k+1)r-12k}{2k+1}\rangle, if\ n=6k+1\\ \langle 0,1,2k,1,\frac{(3k+1)r-(3k-1)}{(6k+2)r-(15k+4)}\rangle, if\ n=6k+2\\ \langle 0,1,2k,1,\frac{(4k+2)r-(6k+1)}{(2k+1)(r-3)}\rangle, if\ n=6k+3, k=1,2,3,4..\end{cases}$$

Proof:

$$\frac{PN(n,r)}{PN(n+1,r)}=\frac{n(n+1)\big(n(r-2)-(r-5)\big)/6}{(n+1)(n+2)\big((n+1)(r-2)-(r-5)\big)/6}$$

$$=\frac{n(nr-2n-r+5)}{(n+2)(nr+r-2n-2-r+5)}$$

$$=\frac{n(nr-2n-r+5)}{(n+2)(nr-2n+3)}$$

$$= \frac{n^2r - 2n^2 - nr + 5n}{n^2r - 2n^2 + 3n + 2nr - 4n + 6}$$

$$\frac{PN(n,r)}{PN(n+1,r)} = \frac{n^2r - 2n^2 - nr + 5n}{n^2r - 2n^2 + 2nr - n + 6}$$

Case (i) Take $n = 6k - 2$

$$\frac{PN(6k-2,r)}{PN(6k-1,r)} = \frac{(6k-2)^2r - 2(6k-2)^2 - (6k-2)r + 5(6k-2)}{(6k-2)^2r - 2(6k-2)^2 + 2r(6k-2) - (6k-2) + 6}$$

$$= \frac{36k^2r + 4r - 24kr - 72k^2 - 8 + 48k - 6kr + 2r + 30k - 10}{36k^2r + 4r - 24kr - 72k^2 - 8 + 48k + 12kr - 4r - 6k + 8}$$

$$= \frac{36k^2r - 72k^2 - 30kr + 6r + 78k - 18}{36k^2r - 72k^2 - 12kr + 42k}$$

$$= 0 + \frac{1}{\frac{36k^2r - 72k^2 - 12kr + 42k}{36k^2r - 72k^2 - 30kr + 78k + 6r - 18}}$$

$$= 0 + \frac{1}{1 + \frac{1}{\frac{36k^2r - 72k^2 - 30kr + 78k + 6r - 18}{18kr - 36k - 6r + 18}}}$$

$$= 0 + \frac{1}{1 + \frac{1}{(2k-1) + \frac{6k}{18kr - 36k - 6r + 18}}}$$

$$= 0 + \frac{1}{1 + \frac{1}{(2k-1) + \frac{1}{\frac{(3k-1)r - 3(2k-1)}{k}}}}$$

$$\frac{PN(6k-2,r)}{PN(6k-1,r)} = \langle 0, 1, 2k-1, \frac{(3k-1)r - 3(2k-1)}{k} \rangle$$

2.2 Illustrations

Value	Ratios	Continued fraction
$k = 1$	$\frac{PN(4,r)}{PN(5,r)}$	$\langle 0, 1, 1, \frac{2r-3}{1} \rangle$
$k = 2$	$\frac{PN(10,r)}{PN(11,r)}$	$\langle 0, 1, 3, \frac{5r-9}{2} \rangle$

Case (ii) Take $n = 6k - 1$

$$\frac{PN(6k-1,r)}{PN(6k,r)} = \frac{(6k-1)^2r - 2(6k-1)^2 - (6k-1)r + 5(6k-1)}{(6k-1)^2r - 2(6k-1)^2 + 2r(6k-1) - (6k-1) + 6}$$

$$= \frac{(36k^2 + 1 - 12k)r - 2(36k^2 + 1 - 12k) - 6kr + r + 30k - 5}{(36k^2 + 1 - 12k)r - 2(36k^2 + 1 - 12k) + 12kr - 2r - 6k + 1 + 6}$$

$$= \frac{36k^2r + r - 12kr - 72k^2 - 2 + 24k - 6kr + r + 30k - 5}{36k^2r + r - 12kr - 72k^2 - 2 + 24k + 12kr - 2r - 6k + 7}$$

$$= \frac{36k^2r - 72k^2 - 18kr + 2r + 54k - 7}{36k^2r - 72k^2 - r + 18k + 5}$$

$$= 0 + \frac{1}{\frac{36k^2r - 72k^2 - r + 18k + 5}{36k^2r - 72k^2 - 18kr + 2r + 54k - 7}}$$

$$= 0 + \frac{1}{1 + \frac{18kr - 3r - 36k + 12}{36k^2r - 72k^2 - 18kr + 2r + 54k - 7}}$$

$$= 0 + \frac{1}{1 + \frac{1}{\frac{36k^2r - 72k^2 - 18kr + 2r + 54k - 7}{18kr - 3r - 36k + 12}}}$$

$$= 0 + \frac{1}{1 + \frac{1}{(2k-1) + \frac{6kr - r - 6k + 5}{18kr - 3r - 36k + 12}}}$$

$$= 0 + \frac{1}{1 + \frac{1}{(2k-1) + \frac{1}{\frac{18kr - 3r - 36k + 12}{6kr - r - 6k + 5}}}}$$

$$= 0 + \frac{1}{1 + \frac{1}{(2k-1) + \frac{1}{1 + \frac{1}{\frac{6kr - r - 6k + 5}{12kr - 2r - 30k + 7}}}}}$$

$$\frac{PN(6k-1,r)}{PN(6k,r)} = \langle 0, 1, 2k-1, 1, \frac{(6k-1)r - (6k-5)}{2r(6k-1) - \left(4(6k-1) + 3(2k-1)\right)} \rangle$$

2.3 Illustrations:

Value	Ratios	Continued fraction
$k = 1$	$\frac{PN(5,r)}{PN(6,r)}$	$\langle 0, 1, 1, 1, \frac{5r-1}{10r-23} \rangle$
$k = 2$	$\frac{PN(11,r)}{PN(12,r)}$	$\langle 0, 1, 3, 1, \frac{11r-7}{22r-53} \rangle$

Case (iii) Take $n = 6k$

$$\frac{PN(6k,r)}{PN(6k+1,r)} = \frac{(6k)^2r - 2(6k)^2 - 6kr + 5(6k)}{(6k)^2r - 2(6k)^2 + 2r(6k) - 6k + 6}$$

$$= \frac{36k^2r - 72k^2 - 6kr + 30k}{36k^2 - 72k^2 + 12kr - 6k + 6}$$

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$$= 0 + \frac{1}{\frac{36k^2-72k^2+12kr-6k+6}{36k^2r-72k^2-6kr+30k}}$$

$$= 0 + \frac{1}{1+\frac{18kr-36k+6}{36k^2r-72k^2-6kr+30k}}$$

$$= 0 + \frac{1}{1+\frac{1}{\frac{36k^2r-72k^2-6kr+30k}{18kr-36k+6}}}$$

$$= 0 + \frac{1}{1+\frac{1}{(2k-1)+\frac{12kr-18k+6}{18kr-36k+6}}}$$

$$= 0 + \frac{1}{1+\frac{1}{(2k-1)+\frac{1}{\frac{3kr-6k+1}{2kr-3k+1}}}}$$

$$= 0 + \frac{1}{1+\frac{1}{(2k-1)+\frac{1}{1+\frac{kr-3k}{2kr-3k+1}}}}$$

$$= 0 + \frac{1}{1+\frac{1}{(2k-1)+\frac{1}{1+\frac{1}{\frac{2kr-(3k-1)}{k(r-3)}}}}}$$

$$\frac{PN(6k,r)}{PN(6k+1,r)} = \langle 0, 1, 2k-1, 1, \frac{2kr-(3k-1)}{k(r-3)} \rangle$$

2.4 Illustrations

Value	Ratios	Continued fraction
$k = 1$	$\frac{PN(6,r)}{PN(7,r)}$	$\langle 0, 1, 1, 1, \frac{2r-2}{1(r-3)} \rangle$
$k = 2$	$\frac{PN(12,r)}{PN(13,r)}$	$\langle 0, 1, 3, 1, \frac{4r-5}{2(r-3)} \rangle$

Case (iv) Take $n = 6k + 1$

$$\frac{PN(6k+1,r)}{PN(6k+2,r)} = \frac{(6k+1)^2r - 2(6k+1)^2 - (6k+1)r + 5(6k+1)}{(6k+1)^2r - 2(6k+1)^2 + 2r(6k+1) - (6k+1) + 6}$$

$$= \frac{36k^2r + r + 12kr - 72k^2 - 2 - 24k - 6kr - r + 30k + 5}{36k^2r + r + 12kr - 72k^2 - 2 - 24k + 12kr + 2r - 6k + 5}$$

$$= \frac{36k^2r - 72k^2 + 6kr + 6k + 3}{36k^2r - 72k^2 + 24kr + 3r - 30k + 3}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{\frac{36k^2r - 72k^2 + 6kr + 6k + 3}{3r + 18kr - 36k}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{2k + \cfrac{1}{\frac{18kr - 36k + 3r}{6k+3}}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{2k + \cfrac{1}{\frac{(6k+1)r - 12k}{2k+1}}}}$$

$$\frac{PN(6k+1,r)}{PN(6k+2,r)} = \langle 0, 1, 2k, \frac{(6k+1)r - 12k}{2k+1} \rangle$$

2.5 Illustrations :

Value	Ratios	Continued fraction
$k = 1$	$\frac{PN(7,r)}{PN(8,r)}$	$\langle 0, 1, 2, \frac{7r - 12}{3} \rangle$
$k = 2$	$\frac{PN(13,r)}{PN(14,r)}$	$\langle 0, 1, 4, \frac{13r - 24}{5} \rangle$

Case (v) Take $n = 6k + 2$

$$\frac{PN(6k+2,r)}{PN(6k+3,r)} = \frac{(6k+2)^2r - 2(6k+2)^2 - (6k+2)r + 5(6k+2)}{(6k+2)^2r - 2(6k+2)^2 + 2r(6k+2) - (6k+2) + 6}$$

$$= \frac{(36k^2 + 4 + 24k)r - 2(36k^2 + 4 + 24k) - 6kr - 2r + 30k + 10}{(36k^2 + 4 + 24k)r - 2(36k^2 + 4 + 24k) + 12kr + 4r - 6k - 2 + 6}$$

$$= 0 + \cfrac{1}{\frac{36k^2r - 72k^2 + 36kr + 8r - 54k - 4}{36k^2r - 72k^2 + 18kr + 2r - 18k + 2}}$$

$$= 0 + \cfrac{1}{1 + \frac{18kr + 6r - 36k - 6}{36k^2r - 72k^2 + 18kr + 2r - 18k + 2}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{\frac{36k^2r - 72k^2 + 18kr + 2r - 18k + 2}{18kr + 6r - 36k - 6}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{2k + \frac{6kr + 2r - 6k + 2}{18kr + 6r - 36k - 6}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{2k + \cfrac{1}{1 + \cfrac{1}{\frac{6kr + 2r - 6k + 2}{12kr + 4r - 30k - 8}}}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{2k + \cfrac{1}{1 + \cfrac{1}{\frac{3kr + r - 3k + 1}{6kr + 2r - 15k - 4}}}}}$$

$$\frac{PN(6k+2,r)}{PN(6k+3,r)} = \left\langle 0, 1, 2k, 1, \frac{(3k+1)r - (3k-1)}{(6k+2)r - (15k+4)} \right\rangle$$

2.6 Illustrations :

Value	Ratios	Continued fraction
$k = 1$	$\frac{PN(8,r)}{PN(9,r)}$	$\left\langle 0,1,2,1,\frac{4r-2}{8r-19} \right\rangle$
$k = 2$	$\frac{PN(14,r)}{PN(15,r)}$	$\left\langle 0,1,4,1,\frac{7r-5}{14r-34} \right\rangle$

(vi) Take $n = 6k + 3$

$$\frac{PN(6k+3,r)}{PN(6k+4,r)} = \frac{(6k+3)^2 r - 2(6k+3)^2 - (6k+3)r + 5(6k+3)}{(6k+3)^2 r - 2(6k+3)^2 + 2r(6k+3) - (6k+3) + 6}$$

$$= \frac{36k^2 r - 72k^2 + 30kr + 6r - 42k - 3}{36k^2 r - 72k^2 + 48kr + 15r - 78k - 15}$$

$$= 0 + \cfrac{1}{\frac{36k^2r - 72k^2 + 48kr + 15r - 78k - 15}{36k^2r - 72k^2 + 30kr + 6r - 42k - 3}}$$

$$= 0 + \cfrac{1}{1 + \frac{18kr + 9r - 36k - 12}{36k^2r - 72k^2 + 30kr + 6r - 42k - 3}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{\frac{36k^2r - 72k^2 + 30kr + 6r - 42k - 3}{18kr + 9r - 36k - 12}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{2k + \cfrac{1}{\frac{18kr + 9r - 36k - 12}{12kr + 6r - 18k - 3}}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{2k + \cfrac{1}{1 + \cfrac{1}{\frac{12kr + 6r - 18k - 3}{6kr + 3r - 18k - 9}}}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{2k + \cfrac{1}{1 + \cfrac{1}{\frac{4kr + 2r - 6k - 1}{2kr + r - 6k - 3}}}}}$$

$$\frac{PN(6k+3,r)}{PN(6k+4,r)} = \langle 0,1,2k,1,\frac{(4k+2)r-(6k+1)}{(2k+1)(r-3)}\rangle$$

2.7 Illustrations :

Value	Ratios	Continued fraction
$k=1$	$\frac{PN(9,r)}{PN(10,r)}$	$\langle 0,1,2,1,\frac{6r-7}{3(r-3)}\rangle$
$k=2$	$\frac{PN(15,r)}{PN(16,r)}$	$\langle 0,1,4,1,\frac{10r-13}{5(r-3)}\rangle$

2.8 Theorem :
If $'n'$ and $'r'$ are order and rank of the corresponding pyramidal numbers then the continued function of

$$\frac{PN(n,r)}{PN(n,r+1)} = \langle 0,1,(r-2),\frac{n-1}{3}\rangle$$

Proof:

$$\frac{PN(n,r)}{PN(n,r+1)} = \frac{\left(n(n+1)\left(n(r-2)-(r-5)\right)\right)/6}{\left(n(n+1)\left(n(r+1-2)-(r+1-5)\right)\right)/6}$$

$$= 0+\frac{1}{\frac{nr-n-r+4}{nr-2n-r+5}}$$

$$= 0+\frac{1}{1+\frac{1}{(r-2)+\frac{1}{\frac{n-1}{3}}}}$$

$$\frac{PN(n,r)}{PN(n,r+1)} = \langle 0,1,(r-2),\frac{n-1}{3}\rangle$$

2.9 Illustrations
2.10

Ratios	Continued fraction
$\frac{PN(n,3)}{PN(n,4)}$	$\langle 0,1,1,\frac{n-1}{3}\rangle$
$\frac{PN(n,5)}{PN(n,6)}$	$\langle 0,1,3,\frac{n-1}{3}\rangle$

3. Conclusion

Continued fractions of pyramidal numbers with higher orders in the numerator are studied here. When the ratios of continued fractions with consecutive ranks are taken six cases arise whenever for consecutive ordered fractions only a single case arise. Consecutive ordered pyramidal numbers have been taken up for study.

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