

Stability of the optimal schedule for perishable product processing

*Albert I. Egamov**

Lobachevsky University of Nizhny Novgorod, 23, Gagarin ave. Nizhny Novgorod, Russian Federation

Abstract. In linear programming tasks, the optimal solution can remain the same even with a significant deviation of the initial data. Thus, when studying various problems in economic and mathematical analysis, the question of the optimal solution stability often arises. The problem of finding the optimal schedule for processing perishable products is solved below. For example, we could refer to sugar beet, an important strategic product that degrades during storage, losing sucrose over time according to some law, depending on time and (or) variety. Other things being equal, with an increase in incoming sucrose, the yield of the final product - sugar, also increases, therefore, by maximizing incoming sucrose, it is possible to significantly increase the production profitability. In practice, equipment often breaks down, therefore, the processing of raw materials stops for a while, but not its degradation. In this regard, the optimal schedule after the production resumption may change, or it may remain the same. Definitions of the optimal schedule stability are given. It is proved that the optimal schedules for the main special cases are absolutely stable. Examples of conditional stability and local stability for a period are given, as well as a numerical experiment showing averaged absolute and relative losses for various parameters of raw materials batches and various periods of production stoppage.

1 Introduction

The sustainability concept finds its application and interpretation in many scientific fields, for example, in mathematics, physics, biology, architecture, agriculture, and many others. In technology, stability is the ability of a system to return to its original state after external effects and continue to work without changing functional characteristics [1]. In this article, the concept of optimal schedule stability is introduced and analyzed, that is, the preservation of the optimal schedule after an external negative impact: equipment breakdowns, power outages, etc. The production schedule is the most important part of the operational and strategic planning system for the functioning of an industrial enterprise, since it affects almost all aspects of its activities. At the same time, a scientifically based optimal production schedule provides a significant increase in enterprise efficiency [2].

* Corresponding author: albert810@yandex.ru

Unlike "continuous" mathematics, where a small change in a parameter usually entails a change in the optimal solution, in discrete optimization, even significant changes in the objective function, constraint parameters, etc., can leave the optimal solution or optimal order unchanged. Therefore, a number of articles in this area are devoted to the stability of optimal control or optimal control sensitivity with respect to changes in parameters, especially when describing economic models related to the agro-industrial complex, where they lead to some optimization problems of linear programming [3-5].

A number of recently studied problems in which the stability of optimal solutions is considered are presented in [6-9]. Sensitivity in optimization is the subject of a monograph [10], the author of which believes that the concept of "sensitivity" is related, but a deeper concept than the concept of "stability", since it can be interpreted as quantitative stability.

The interval assignment problem is considered in [11, 12]. There in, the components of the cost matrix are the acceptable segments to which the corresponding parameters belong, and the question of optimal control stability is also raised. Various types of stability in linear programming are described in [13].

The handbook [14] states that "... the economic and mathematical analysis of the solution for optimization problems consists primarily in identifying the conditions under which the obtained solution of the problem is stable, i.e. the found plan remains optimal with relatively small changes in the initial conditions. For this purpose, a number of more or less similar variants of the task are calculated and compared...".

The limits of parameter variation of the objective function or by the parameters of one or more constraints so that the optimal schedule remains the same are discussed in the works of specialists on this topic. The problem obtained in this paper belongs to the class of discrete optimization problems and is a special case of the well-known "assignment problem". To solve it, either the Hungarian algorithm is used, or similar ones, for example, the Mack's algorithm [15], [16], or the auction algorithm [17], it is also possible to reduce the assignment problem to the minimum-cost flow problem [18]. In [19-22], a mathematical model for constructing an optimal processing schedule for n batches of perishable raw materials was set and studied. In these works, sugar beet is meant as a raw material, which is harvested during the ripening period and stored at beet pile fields during the season to ensure further continuous production process. A useful ingredient is sugar, during storage the sugar content of beets decreases [23]. In addition to the optimal schedule, these works touched on the topic of a quasi-optimal schedule, which is obtained from the optimal one with some variation in the parameters of sugar content and batch degradation. In this article, the mathematical model of product processing is extended. It is assumed that equipment may break down in production and a certain period of time will be spent on its repair. During the repair period, the proportion of the useful ingredient, and therefore the optimal schedule may change. Especially interesting are the cases when the optimal schedule after the resumption of work remains the same or, at least, in the optimal schedule, the batch that goes for processing immediately after the production shutdown will not change. The article will present various definitions of the optimal schedule stability and examples of their existence. The proof of the absolute stability of optimal schedules is also carried out for two main special cases, when the degradation coefficients of batches depend on one argument only out of two. A computational experiment has been set up that gives an answer to the question of the strategy quantitative loss, according to which, after the shutdown, the production schedule (optimal before the shutdown) remains unchanged.

2 Task statement

2.1 Mathematical model of processing

Similarly to the works [19-22], we describe a mathematical model of sugar beet processing. Let there be n lots of equal mass, numbered from 1 to n . The mass of one batch is the mass that the production facilities of the enterprise can process in 1 stage (period) – a certain period of time (for example, in one day or one week). Without limiting generality, we can assume that $M = 1$. Different batches differ in production value, which corresponds to the percentage of useful ingredient content in the corresponding batch. We denote $a_i, i = \overline{1, n}$ – production value of the i batch at the beginning of the first processing period. To process n batches, it is necessary to have n stages, we number them from 1 to n . We assume that during storage at the j processing stage, the i batch loses some of its production value (for example, beetroot reduces its sugar content). We denote b_{ij} – degradation coefficient, which determines the decrease in the share of the useful ingredient of the i batch at the j stage, $i = \overline{1, n}, j = \overline{1, n - 1}$, during one stage of processing of this batch, its production value does not change. We build a matrix P of the order $n \times n$ with components p_{ij} , where p_{ij} is the production value of the i batch before the j stage:

$$p_{i1} = a_i, \quad p_{ij} = a_i b_{i1} b_{i2} \dots b_{ij-1}. \tag{1}$$

Let the order of processing batches be given by a permutation σ , where $\sigma = \begin{pmatrix} 1 & 2 & \dots & j & \dots & n-1 & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(j) & \dots & \sigma(n-1) & \sigma(n) \end{pmatrix}$ is some permutation of natural numbers from 1 to n , then the objective function has the form:

$$S = p_{\sigma(1)1} + p_{\sigma(2)2} + p_{\sigma(3)3} + \dots + p_{\sigma(n)n}. \tag{2}$$

It is clear that if the permutation σ sets the order of the batches, which provides for the function S the maximum value S^* , and $S_{k \rightarrow m}$ is the value of the objective function (2) when rearranging the places in the optimal schedule of the k and m batches, then this means that the permutation σ sets the optimal schedule and the inequality is satisfied:

$$S^* - S_{k \rightarrow m} \geq 0. \tag{3}$$

Such inequalities resulting from the permutation of some two lines in the optimal schedule, we call the necessary optimality conditions.

The objective function (2) according to (1) can be rewritten in the equivalent form:

$$S = a_{\sigma(1)} + a_{\sigma(2)} b_{\sigma(2)1} + a_{\sigma(3)} b_{\sigma(3)1} b_{\sigma(3)2} + \dots + a_{\sigma(n)} b_{\sigma(n)1} b_{\sigma(n)2} \dots b_{\sigma(n)n-1}. \tag{4}$$

The inequality (3) according to (1) can be rewritten in the equivalent form:

$$a_{\sigma(k)} b_{\sigma(k)1} \cdot \dots \cdot b_{\sigma(k)k-1} \left(1 - \prod_{j=k}^{m-1} b_{\sigma(k)j} \right) \geq a_{\sigma(m)} b_{\sigma(m)1} \cdot \dots \cdot b_{\sigma(m)k-1} \left(1 - \prod_{j=k}^{m-1} b_{\sigma(m)j} \right). \tag{5}$$

The task of plotting an optimal processing schedule in general consists in choosing such a sequence of processing of raw materials, given by a permutation σ from 1 to n , for which the value S will be maximum. We assume that the final product output after the completion of all stages will be proportional to the value of the objective function – the proportion of incoming sucrose, therefore it is necessary to maximize the objective function (1) or the

equivalent objective function (4). In total, there are $n!$ various permutations (different processing schedules) of n sugar beet batches, that is, when solving the problem by the bruce force method, it is necessary to calculate and compare the $n!$ values of the objective function. Nevertheless, there are other, more effective solution methods.

The task of maximizing the function S in the general case can be reduced to the well-known "assignment problem", for the solution of which an algorithm has been built, called the "Hungarian algorithm" with time complexity $O(n^4)$. This algorithm can find both the maximum and minimum of the objective function, as well as the corresponding permutations of the batches – the corresponding processing orders of the batches. Nevertheless, the Hungarian algorithm and its analogues can be applied only if the matrix P is fully known in advance (before the start of processing the first batch) or all the details of the process of production value loss by all batches of raw materials during storage are known, that is, all $a_i, i = \overline{1, n}$, and $b_{ij}, i = \overline{1, n}, j = \overline{1, n - 1}$, are known that it is practically possible to implement only empirically. Thus, in practice, it is often more justified for optimization tasks to use understandable and reasonable quasi-optimal solutions based on some estimates of the degradation of raw materials batches. One of the ways to solve the problem in practice is to present the problem in the form of the interval assignment problem. A different approach has been used in a number of studies [19-22] devoted to the issues of drawing up optimal and quasi-optimal processing schedules.

2.2 Optimal schedule stability

The mathematical model described above does not consider the case of a plant shutdown due to a breakdown. In real conditions, the production line breakdown can occur at any time and lasts for several periods. We will assume that during the entire processing, the breakdown was once and lasted for one period. To consider this, it is necessary to expand the mathematical model. Suppose that during the J period, $J = \overline{2, n - 1}$, production was idle, but starting from the $(J + 1)$ period, it continued to work. For model correctness, it is necessary to determine the degradation coefficients $b_{in}, i = \overline{1, n}$, during the n period and p_{in+1} , since the processing now lasts $(n + 1)$ period. We call the optimal processing schedule conditionally stable in the period J , if the optimal schedule remains optimal considering the new operating mode up to $(n + 1)$ period inclusively and the new degradation coefficients $b_{in}, i = \overline{1, n}$, , otherwise we consider it unstable for the period J .

In other words, let the permutation σ set the optimal schedule (batch processing order), which does not consider the production shutdown, the new optimal processing schedule is set by the permutation $\tilde{\sigma}$, then the permutations are identical, that is, if $\tilde{\sigma}(i) = \sigma(i), i = \overline{1, n}$, then the optimal schedule remains the same.

Naturally, it is impossible to change the optimal processing schedule from the 1 to the $(J - 1)$ time period, the time has already passed and processing during these periods has already been carried out (initially it is not known when the breakdown will happen).

If the stability condition is met for any acceptable J , then the optimal schedule is absolutely stable. If after the shutdown and resumption of work, according to the new optimal schedule, the same batch that would have been processed during the J period under the old optimal schedule will be processed, then the optimal schedule is locally stable for the J period. It is not difficult to see that from absolute stability conditional stability for the period J follows, as well as from conditional stability for the period J local stability for the period J follows. Examples of the existence of all declared types of stability of the optimal schedule will be shown below.

3 Main results

3.1 Changing the objective function after shutdown

Let there be a production stoppage in the J period, batches with numbers $1 - (J - 1)$ have been processed before it. In fact, after the resumption of production, the task of finding the optimal schedule is reduced to a similar one, but for the remaining $n - J + 1$ batches with "initial" sugar content: $p_{(J+m)J}$, $m = \overline{0, (n - J)}$.

We will number the batches in the order of processing. Considering the shutdown of production during the period J , the objective function (4) takes the form \tilde{S} :

$$\tilde{S} = a_1 + a_2 b_{21} + \dots + a_{J-1} b_{J-1,1} \cdot \dots \cdot b_{J-1, J-2} + a_J b_{J1} \dots b_{JJ} + \dots + a_n b_{n1} \cdot \dots \cdot b_{nn}. \quad (6)$$

Denote $B_{ij} = b_{i1} b_{i2} \dots b_{ij}$, $i = \overline{1, n}$, $j = \overline{1, n}$, then as $p_{ij} = a_i B_{ij-1}$, and expressions (4), (6) will be rewritten as

$$S = a_1 + a_{21} B_{21} + \dots + a_{J-1} B_{J-1, J-2} + a_J B_{JJ-1} + \dots + a_n B_{nn-1}, \quad (7)$$

$$\tilde{S} = a_1 + a_{21} B_{21} + \dots + a_{J-1} B_{J-1, J-2} + a_J B_{JJ} + \dots + a_n B_{nn} \quad (8)$$

accordingly. Denote \tilde{S}^* – the optimal (maximum) value of the objective function (8), considering that shutdown occurred during the J period.

For clarity, we will first show examples of absolute stability for two important special cases. The first case is when the degradation coefficients do not depend on the batch number (grade), but depend only on the period (time), the second is when the degradation coefficients do not depend on the time (period), but depend only on the batch number (grade).

3.2 The first case of the absolutely stable optimal schedule

In the future, we will need a permutation inequality [24].

Permutation inequality.

Let $u = (u_1, u_2, \dots, u_n)$ and $w = (w_1, w_2, \dots, w_n)$ be two finite sequences of real numbers that satisfy the following conditions: $u_1 \geq u_2 \geq \dots \geq u_n$, $w_1 \geq w_2 \geq \dots \geq w_n$, then the inequalities are valid

$$u_1 w_n + u_2 w_{n-1} + \dots + u_n w_1 \leq u_1 w_{\sigma(1)} + u_2 w_{\sigma(2)} + \dots + u_n w_{\sigma(n)} \leq u_1 w_1 + u_2 w_2 + \dots + u_n w_n,$$

here σ – some arbitrary permutation of natural numbers from 1 to n .

One of the important special cases: all batches degrade in the same way, for example, because of the weather, that is, conditions are imposed on the parameters b_{ij} :

$$b_{ij} = \bar{b}_j, \quad i = \overline{1, n}, \quad j = \overline{1, n}, \quad (9)$$

which means the degradation coefficients do not depend on the batch, but depend only on the processing period (time). We formulate the statement in the form of a theorem.

Theorem 1. Let the equalities (9) be true, then the optimal schedule for this case is absolutely stable.

Proof. Without loss generality, assume that all a_i , $i = \overline{1, n}$ are different. Denote $B_j = \prod_{k=1}^j \bar{b}_k$, $j = \overline{1, n}$. It is clear that the following two chains of inequalities are true:

$$1 = B_0 > B_1 > \dots > B_{n-1} \quad (10)$$

and

$$1 = B_0 > B_1 > \dots > B_{j-2} > B_j > \dots > B_n. \tag{11}$$

Variables $B_j, j = \overline{1, n}$, and chains of inequalities (10), (11) do not depend on batch numbers and on the order of their processing.

The objective function (7) will be rewritten as

$$S = a_1 + a_2 B_1 + \dots + a_{j-1} B_{j-2} + a_j B_{j-1} + \dots + a_n B_{n-1}. \tag{12}$$

The objective function (8) with the same schedule after stopping takes the form

$$\tilde{S} = \sum_{i=1}^{j-1} a_i B_{i-1} + \sum_{i=j}^n a_i B_i. \tag{13}$$

Let η be a permutation of numbers from 1 to n , for which inequalities are satisfied

$$a_{\eta(1)} > a_{\eta(2)} > \dots > a_{\eta(n)}. \tag{14}$$

It follows from the permutation inequality that the objective function S (see (12)) is maximal if it consists of the sum of pairwise products of two lines (10) and (14) and has the form:

$$S^* = \sum_{i=1}^n a_{\eta(i)} B_{i-1}.$$

The schedule given by the permutation η is optimal.

Similarly, the objective function (13) takes the maximum value if it consists of the sum of the pairwise products of two lines (11) and (14),

$$\tilde{S}^* = a_{\eta(1)} + a_{\eta(2)} B_1 + \dots + a_{\eta(j-1)} B_{j-2} + a_{\eta(j)} B_j + \dots + a_{\eta(n)} B_n,$$

that is, when rearranging η . Therefore, the permutation will η remain the optimal schedule even after work is stopped, which means that when conditions (9) are met, the optimal schedule is absolutely stable.

3.3 The second case of the absolutely stable optimal schedule

We consider another important special case when the coefficients b_{ij} depend only on the raw material batch number, that is

$$b_{ij} = b_i, i = \overline{1, n}, j = \overline{1, n}. \tag{15}$$

Without limiting generality, assume that all b_i are different. Denote

$$\chi_{0n} = \sqrt{\frac{n-2}{n}}; \quad \chi_n = \frac{n-1}{n}, \quad n \in N.$$

It is not difficult to verify that $\chi_{n-1} < \chi_{0n} < \chi_n$.

Theorem 2. Let the $a_i = a, i = \overline{1, n}$. Equalities be true (15). We number the batches in ascending order b_i . Suppose there is limitation

$$\min_{i=1, n} b_i \geq \chi_{n-1}, \tag{16}$$

and also one of two limitations:

$$1) \quad b_{n-1} \geq \chi_n, \tag{17}$$

then the optimal schedule is absolutely stable;

$$2) \quad \chi_{0n} \leq b_{n-2} < b_{n-1} < b_n \leq \chi_n, \tag{18}$$

then the optimal schedule is not stable, but there will be local stability at $J = \overline{2, n - 2}$.

Proof. The paper [19] considers the case described in the theorem 2 without limitations (17) and (18) on the degradation coefficient b_n . It proves that the maximum value of the objective function for n batches can be obtained if batches of raw materials are processed in ascending order of coefficients b_n . The batch numbering accepted in the theorem is the optimal schedule for the case of n batches and n processing periods. The function (2) considering will be (15) written as

$$S^* = a + ab_2 + ab_3^2 + \dots + ab_n^{n-1}.$$

The objective function (8), considering the shutdown and reaching the maximum will be rewritten as follows:

$$\tilde{S}^* = \sum_{i=1}^{J-1} ab_{\gamma(i)}^{i-1} + \sum_{i=J}^n ab_{\gamma(i)}^i, \tag{19}$$

where γ is the permutation of the optimal schedule after the shutdown.

We consider the processing order, which differs from the optimal one by rearranging $(k - 1)$ and k batches of raw materials, where $k - 1 \geq J$, that is, both batches are processed after shutdown. In this case, the necessary optimality condition will be met for the objective function (19), inequality (5) will be rewritten as

$$\begin{aligned} \tilde{S}^* - \tilde{S}_{(k-1) \rightarrow k} &= \sum_{i=1}^{J-1} ab_{\gamma(i)}^{i-1} + ab_{\gamma(J)}^J \dots + ab_{\gamma(k-1)}^{k-1} + \\ &+ ab_{\gamma(k)}^k + \dots + ab_{\gamma(n)}^n - \sum_{i=1}^{J-1} a_{\gamma(i)} b_{\gamma(i)}^{i-1} - ab_{\gamma(J)}^J - \dots - ab_{\gamma(k)}^{k-1} - \\ &- ab_{\gamma(k-1)}^k - \dots - ab_{\gamma(n)}^n \geq 0, \end{aligned}$$

from which the equivalent inequalities follow

$$\begin{aligned} b_{\gamma(k-1)}^{k-1} + b_{\gamma(k)}^k &\geq b_{\gamma(k)}^{k-1} + b_{\gamma(k-1)}^k \Leftrightarrow \\ &\Leftrightarrow f(b_{\gamma(k-1)}) \geq f(b_{\gamma(k)}), \end{aligned} \tag{20}$$

where the function $f(x)$ has the form $f(x) = x^{k-1} - x^k$, then $f'(x) = (k - 1)x^{k-2} - kx^{k-1}$. It turns to zero when $x^* = \chi_k \leq \chi_n$.

For both clauses of the theorem 2, when $k = \overline{3, n - 1}$, $k \geq J + 1$ (see(16)), on the interval $x \in (x^*, 1)$ the derivative $f'(x)$ is negative, hence the function $f(x)$ monotonically decreases; a smaller value of the function corresponds to a larger value of the argument. It follows from the inequality (20), that $b_{\gamma(k-1)} \leq b_{\gamma(k)}$ for $k = \overline{J + 1, n - 1}$. Therefore, a chain of inequalities is valid for the optimal schedule after production stops.

$$b_{\gamma(k-1)} \leq b_{\gamma(k)} < \dots < b_{\gamma(n-1)}.$$

What inequality should there be between the parameters $b_{\gamma(n-1)}$ and $b_{\gamma(n)}$?

For the clause 1 of the theorem 2, reasoning similarly, we get that $b_{\gamma(n-1)} \leq b_{\gamma(n)}$, which means also $\gamma(i) = i$, $i = \overline{1, n}$, that is, the initial optimal schedule is absolutely stable.

Next we consider the conditions of clause 2. For it, both parameters $b_{\gamma(n-1)}$ and $b_{\gamma(n)}$ belong to a half-interval $(\chi_{n-1}, \chi_n]$, the derivative $f'(x)$ is positive on it, and the function $f(x)$ increases monotonically; a larger value of the function corresponds to a larger value of the argument, and inequality $b_{\gamma(n-1)} \geq b_{\gamma(n)}$ follows from inequality (20), that is, in the new optimal schedule, the equalities $\gamma(n - 1) = n - 1$ and $\gamma(n) = n$ are not satisfied simultaneously, therefore the initial optimal schedule is unstable for any acceptable J .

We will consider the processing order, which differs from the optimal schedule given by the γ , permutation of the $(n - 2)$ and n batches of raw materials. $J \leq n - 2$ (that is, both

batches are processed after the shutdown). An inequality analog (5) will be performed for the objective function:

$$\begin{aligned} 0 &\leq \tilde{S}^* - \tilde{S}_{\gamma(n-2) \rightarrow \gamma(n)}^* = \\ &= a + \dots + ab_{\gamma(n-3)}^{n-3} + ab_{\gamma(n-2)}^{n-2} + b_{\gamma(n-1)}^{n-1} + b_{\gamma(n)}^n \geq \\ &\geq a + \dots + ab_{\gamma(n-3)}^{n-3} + b_{\gamma(n)}^{n-2} + ab_{\gamma(n-1)}^{n-1} + ab_{\gamma(n-2)}^n, \end{aligned}$$

from which the equivalent inequalities follow

$$\begin{aligned} ab_{\gamma(n-2)}^{n-2} + b_{\gamma(n)}^n &\geq ab_{\gamma(n)}^{n-2} + ab_{\gamma(n-2)}^n \Leftrightarrow \\ \tilde{f}(b_{\gamma(n-2)}) &\geq \tilde{f}(b_{\gamma(n)}), \end{aligned} \tag{21}$$

where the function $\tilde{f}(x)$ has the form $\tilde{f}(x) = x^{n-2} - x^n$. Its first derivative $\tilde{f}'(x) = (n-2)x^{n-3} - nx^{n-1}$ is taken when $x = \chi_0$. By direct substitution, we make sure that when $x > \chi_0$ the inequality $\tilde{f}'(x) < 0$, is true, that is, when $x > \chi_0$ the function $\tilde{f}(x)$ decreases, and, since from the condition of the theorem $b_{\gamma(n-2)}$ and $b_{\gamma(n)}$ are more than χ_0 , is necessary $b_{\gamma(n-2)} \leq b_{\gamma(n)}$ to fulfill the inequality (21).

Thus, the chain of inequalities is true:

$$b_{\gamma(1)} < b_{\gamma(2)} < \dots < b_{\gamma(n-2)} < b_{\gamma(n)} < b_{\gamma(n-1)},$$

which is now determined unambiguously, namely, after the shutdown, the optimal schedule has the form $\{b_1, b_2, b_3, \dots, b_{n-2}, b_n, b_{n-1}\}$. Therefore, when $J = \overline{2, n-2}$ the optimal schedule is locally stable, but is not conditionally stable for any period, , since the last batch in the new optimal schedule does not have the highest coefficient, b_n does not stand in last place.

3.4 Conditionally stable optimal schedule in the J period

We consider an example of conditional stability in the J period. Let there be a production shutdown in the J period,

Theorem 3. *Let the parameters a_i, b_{ij} satisfy the conditions:*

- 1) $b_{ij} = \overline{b_j}, i = \overline{1, n}, j = \overline{3, n}. \tag{22}$
- 2) $a_1 > a_2, a_k = \theta_k a_2, \text{ where } \theta_k, k = \overline{3, n}, \text{ is a finite decreasing sequence, } 0 < \theta_k < 1;$
- 3) *The equalities are valid: $b_{i1} = b_{i2}, i = \overline{1, n}, b_{11} < b_{21}$, and the expressions $b_{k1} = \beta_k b_{21}, \beta_k > 1, k = \overline{3, n}$. where β_k , is a finite increasing sequence.*

In addition, the inequalities are true

$$0 < \beta_k^{-2} < \theta_k < \beta_k^{-1}, k = \overline{3, n}, \tag{23}$$

and equality

$$\theta_3 \beta_3^2 = \max_{k \geq 3} \theta_k \beta_k^2 \tag{24}$$

holds.

Let the production shutdown occurred during the J period, $J = \overline{2, n-1}$.

Then the batches are numbered according to the initial optimal schedule and the optimal processing order found for these parameters will be conditionally stable for time periods J, $J = \overline{3, n-1}$.

Proof. We prove that for an optimal schedule v that does not consider the production shutdown, for parameters satisfying the conditions of the theorem, it is true that $v(i) = i, i = \overline{1, 3}$.

We prove that in the optimal processing schedule, the batch with the number 1 is in 1 place. Let it not be so. What is the count of the first batch that can be processed?

Suppose in the optimal schedule given by the permutation v , the first batch is in the m place, and the k batch is in the first place, that is $v(k) = 1, k \geq 2, v(1) = m = 2$, swap the first and second batches of the optimal schedule v , then according to the necessary optimality condition:

$$S^* - S_{1 \rightarrow 2} = a_{v(k)} + a_1 b_{11} - a_1 - a_{v(k)} b_{v(k)} \geq 0 \Leftrightarrow a_{v(k)}(1 - b_{v(k)1}) \geq a_1(1 - b_{11}), \quad (25)$$

but from the conditions of the theorem $a_{v(k)} < a_1$, when $k \geq 2$, and $(1 - b_{v(1)1}) < (1 - b_{11})$, since $b_{11} < b_{i1}$ for any $i \geq 2$, therefore inequality (25) is not true for any $k \geq 2$, therefore $v(1) \neq 2$.

Let $v(1) = m \geq 3, v(k) = 1, k \geq 2$, swap the first and third batches of the optimal schedule v , then according to the necessary optimality condition:

$$S^* - S_{1 \rightarrow 3} = a_{v(k)} + a_1 b_{11} b_{12} \mu - a_1 - a_{v(k)} b_{v(k)1} b_{v(k)2} \mu \geq 0 \Leftrightarrow a_{v(k)}(1 - b_{v(k)1} b_{v(k)2} \mu) \geq a_1(1 - b_{11} b_{12} \mu), \Leftrightarrow a_{v(k)}(1 - \mu b_{v(k)1}^2) \geq a_1(1 - \mu b_{11}^2), \quad (26)$$

where $\mu = b_{i3} \dots b_{im-1}$ and $m \geq 3$ (μ does not depend on the parameter i , see (22)), if $m = 2$, then $\mu = 1$. Nevertheless, $a_{v(k)} < a_1$, when $k \geq 2$ and $(1 - \mu b_{v(1)1}^2) < (1 - \mu b_{11}^2)$, since $b_{11} < b_{i1}$, for any $i \geq 2$, therefore inequality (26) is not true for any $k \geq 2$, therefore $v(1) < 3$. Given the previous reasoning, we get that $v(1) = 1$.

We prove that $v(2) = 2$. Suppose the opposite: $v(2) = m \geq 3$. Then $k \geq 3$ exists there for which $v(k) = 2$. Let's swap the second and m batches of the optimal schedule v , then according to the necessary optimality condition:

$$\begin{aligned} S^* - S_{2 \rightarrow m} &= a_{v(k)} b_{v(k)1} + a_2 b_{21} b_{22} \mu - a_2 b_{21} - a_{v(k)} b_{v(k)1} b_{v(k)2} \mu \geq 0 \\ &\Leftrightarrow a_{v(k)} b_{v(k)1} (1 - \mu b_{v(k)2}) \geq a_2 b_{21} (1 - \mu b_{22}) \\ &\Leftrightarrow a_{v(k)} b_{v(k)1} (1 - \mu b_{v(k)1}) \geq a_2 b_{21} (1 - \mu b_{22}) \\ &\Leftrightarrow a_{v(k)} b_{v(k)1} (1 - \mu b_{v(k)1}) \geq a_2 b_{21} (1 - \mu b_{21}). \quad (27) \end{aligned}$$

Contradiction, since $1 - \mu b_{v(k)1} < 1 - \mu b_{21}$, since $b_{v(k)1} > b_{21}$, moreover $a_{v(k)} b_{v(k)1} = \theta_{v(k)} \beta_{v(k)} a_2 b_{21} < a_2 b_{21}$, since it follows from (23) that $\theta_{v(k)} \beta_{v(k)} < 1$, then inequality (27) is not true. Contradiction, therefore, $v(2) = 2$.

Further, from the condition (22) according to the theorem 1, processing proceeds according to sugar content descending, which was formed by the third period. For $i \geq 3$ according to condition (24), the sugar content p_{i3} is maximal among the remaining ones, therefore $v(3) = 3$. It follows from the theorem 1 that the processing further proceeds in descending order of numbers p_{i3} and the optimal schedule is absolutely stable, if you do not consider the batches with numbers 1 and 2. That is, when $J = 3, n - 1$ the optimal schedule is conditionally stable.

Next we prove that the optimal schedule is unstable when $J = 2$, that is, it is not absolutely stable. Even if this is not the case, the optimal schedule remains the same. In the optimal schedule, we will swap the second line and the third one, considering the shutdown. Then it follows from the necessary optimality condition that

$$\begin{aligned} \tilde{S}^* - \tilde{S}_{2 \rightarrow 3} &= a_2 b_{21} b_{22} + a_3 b_{31} b_{32} b_{33} - (a_3 b_{31} b_{32} + a_2 b_{21} b_{22} b_{23}) \geq 0 \Leftrightarrow \\ &\Leftrightarrow a_2 b_{21}^2 (1 - b_{23}) \geq a_3 b_{31}^2 (1 - b_{33}) \Leftrightarrow a_2 b_{21}^2 \geq a_3 b_{31}^2 \Leftrightarrow \\ &\Leftrightarrow a_2 b_{21}^2 \geq \theta_3 \beta_3^2 a_2 b_{21}^2 \Leftrightarrow 1 \geq \theta_3 \beta_3^2. \end{aligned}$$

A contradiction, since from inequality (23) it follows that $\theta_3\beta_3^2 > 1$. That is, the necessary optimality condition for the second and third lines is not met. The optimal schedule is unstable when $J = 2$, therefore there is no absolute stability and the optimal processing order is conditionally stable for periods $J, J = 3, n - 1$.

4 Numerical experiment

4.1 Assessment of possible losses for $n = 100$

How much can the objective function lose if, after the shutdown, it uses the previous optimal (but already because of the shutdown – (possibly not optimal) schedule? This question can be answered by conducting numerical experiments using a personal computer. Using the Hungarian algorithm, it is possible to obtain the maximum value of the objective function estimate for any parameters $a_i, b_{ij}, i = \overline{1, n}, j = \overline{1, n}$, if they are all known; or to give an average estimate also numerically, after conducting a sufficient number of experiments (for example, 50), similar to the works [19-22]. To complete the picture, we will conduct 2 series of experiments for $n = 100$ and $n = 20$.

Let $n = 100$ one time period be equal to one day, the series consists of 50 experiments. For each series of experiments, we will set the production shutdown period – a period J , a constant ψ by means of a uniform distribution at random intervals. For each experiment, sets of parameters $a_i, b_{ij}, i = \overline{1, n}, j = \overline{1, n}, a_i \in (0,15,0.25), b_{ij} \in (\psi, 0.99)$ are generated. In each experiment, the following were found:

- 1) the maximum value of the objective function without considering the shutdown – S^* ,
- 2) the maximum value of the objective function, considering the shutdown – \tilde{S}^* ,
- 3) the maximum value of the objective function, considering the shutdown, if processing went according to the old optimal schedule – \tilde{S}_0 ,
- 3) the difference between the values of the objective functions cl. 2 and cl. 3 – $\Delta\tilde{S}_0$, where $\Delta\tilde{S}_0 = \tilde{S}^* - \tilde{S}_0$,
- 4) the averaged values of the objective functions of clauses 1-3 without considering and considering the shutdown,
- 5) relative average losses $\omega, \omega = \frac{\langle \Delta\tilde{S}_0 \rangle}{\langle \tilde{S}^* \rangle} \times 100\%$.

Table 1. Comparison of objective functions with new and old optimal schedules or $b_{ij} \in (\psi, 0.99)$, $n = 100$

J	ψ	$\langle S^* \rangle$	$\langle \tilde{S}^* \rangle$	$\langle \tilde{S}_0 \rangle$	$\langle \Delta\tilde{S}_0 \rangle$	ω
2	0.85	3.395	3.163	3.046	0.117	3.70
25	0.85	3.395	3.361	3.334	0.027	0.80
50	0.85	3.395	3.390	3.385	0.005	0.15
2	0.9	4.573	4.346	4.223	0.123	2.83
25	0.9	4.573	4.516	4.480	0.036	0.80
50	0.9	4.573	4.561	4.556	0.005	0.11

2	0.93	4.591	4.364	4.262	0.102	2.33
25	0.93	4.591	4.534	4.508	0.026	0.57
50	0.93	4.591	4.580	4.574	0.006	0.13
2	0.95	7.296	7.081	7.008	0.073	1.03
2	0.97	9.498	9.298	9.251	0.047	0.50

Table 2. Comparison of objective functions with new and old optimal schedules for $b_{ij} \in (0.99,0.9999)$, $n = 100$.

J	$\langle S^* \rangle$	$\langle \tilde{S}^* \rangle$	$\langle \tilde{S}_0 \rangle$	$\langle \Delta \tilde{S}_0 \rangle$	ω
2	16.268	16.116	16.069	0.047	0.29
25	16.268	16.142	16.107	0.035	0.21
50	16.268	16.165	16.141	0.024	0.15

These results for $\max b_{ij} < 0.99$ are tabulated 1. The table 2 shows the results when the degradation coefficients are very close to 1, $b_{ij} \in (0.99,0.9999)$. It can be seen from table 2 that in this case the losses due to the fact that the schedule was not changed are relatively small.

4.2 Assessment of possible losses for $n = 20$

Let $n = 20$, 1 period be equal to a week. All other actions are similar to numerical experiment 1 (when $n = 100$). The values S^* , \tilde{S}^* , \tilde{S}_0 , $\Delta \tilde{S}_0$, $\langle S^* \rangle$, $\langle \tilde{S}^* \rangle$, $\langle \tilde{S}_0 \rangle$, $\langle \Delta \tilde{S}_0 \rangle$, ω are found. These results are summarized in a table 3, each row corresponds to its own series of experiments.

Table 3. Comparison of objective functions with new and old optimal schedules for $b_{ij} \in (\psi, 0.99)$, $n = 20$.

J	ψ	$\langle S^* \rangle$	$\langle \tilde{S}^* \rangle$	$\langle \tilde{S}_0 \rangle$	$\langle \Delta \tilde{S}_0 \rangle$	ω
2	0.85	2.025	1.863	1.789	0.074	4.00
10	0.85	2.025	1.951	1.926	0.025	1.28
2	0.87	2.156	1.995	1.932	0.063	3.16
10	0.87	2.156	2.074	2.047	0.027	1.30
2	0.9	2.379	2.220	2.167	0.053	2.39
10	0.9	2.379	2.286	2.261	0.025	1.09
2	0.93	2.637	2.484	2.443	0.041	1.65
10	0.93	2.637	2.535	2.512	0.023	0.91
2	0.95	2.833	2.685	2.654	0.031	1.15
10	0.95	2.833	2.725	2.707	0.018	0.66

2	0.97	3.05	2.909	2.891	0.018	0.62
10	0.97	3.05	2.936	2.925	0.011	0.37

It should be noted that S is a proportionality coefficient between the incoming raw materials for the period and the total sugar yield. Under the assumptions made, the sucrose input and, accordingly, the sugar output is equal to $M \cdot S$, where M is the mass of processed raw materials for the period. For example, the Sergach Sugar Factory processes 3,000 tons of sugar beet per day. That is, when $n = 100 - M = 3000$ tons, and for $n = 20 - M = 21000$ tons. For example, the largest difference between the average values of the new and old optimal schedule in Table 1 is 0.117, that is, 351 tons of sugar were not received, and in the third – 0.074, that is, 1554 tons of sugar were not received.

5 Discussion

If all the parameters of the parties are known, then it is not difficult to find out the stability of the optimal schedule in general, it is enough for everyone $J, J = \overline{2, n - 1}$, to calculate a new optimal schedule using the Hungarian algorithm and compare it with the old one. This is "not long" in terms of machine time. Nevertheless, as mentioned above, in practice, it is not possible to know all the exact parameters of the parties a priori. But with the help of computer modeling, it is possible to calculate the average difference between the values of the objective function for the "new" and "old" optimal schedule in a series of experiments with the assumed parameters and then evaluate the losses due to the fact that the original optimal processing schedule has not been replaced.

The sustainability of the optimal schedule is important for the enterprise working on it. If the schedule is absolutely stable, it means that at any time moment there would be a breakdown, the optimal schedule remains the same. This has a positive impact on both long-term planning and logistics, as well as on the working environment in case of force majeure. Conditional stability during the period J means that you can confidently schedule work if production stops during this period of time. Local stability during the period J shows that at least after the breakdown is eliminated, the same batch is being processed that was planned if production had not stopped. There is also a positive moment in this, since raw materials are brought to production in advance.

Numerical experiment has shown that if the number of the shutdown period is not less than half of the total number of batches, then, generally speaking, there is no need to worry about changing the optimal schedule at acceptable intervals fixed in experiments, nevertheless, if the shutdown period is at the beginning of processing, then it is desirable to change the optimal schedule. If the degradation coefficients of the batches b_{ij} for $n = 100$ for the period is at least 0.99 (that is, for example, under very good storage conditions), then it also makes almost no sense to change the optimal schedule.

It is not difficult to make sure that you can also consider a simple production, which lasts not one, but several days. To consider this within the framework of the mathematical model, it is necessary to combine all these days into a certain period and enter for each batch of product a total degradation coefficient for the entire production downtime equal to the product of the degradation coefficients of this batch for the days of downtime. It is also possible to introduce the possibility of not one, but several downtimes into the mathematical model. To do this, it is necessary to change the objective function: put in it only those terms that correspond to working days, that is, when production was working and processing of raw materials was carried out. What is the maximum number of percentages of the objective function for an optimal schedule that can be neglected? How many days of downtime are

there on average per season at the enterprise? What value ψ is more appropriate to put for processed sugar beet hybrids of a particular region? The search for answers to these questions is a subject for further research by specialists in the relevant fields.

6 Conclusion

In this paper, a mathematical model of the optimal schedule for processing a finite number of batches of perishable product has been set and expanded relative to previous works. The expansion of the model consists in the fact that in the J period of time, $J = \overline{2, n-1}$, production stops due to equipment failure (power outages, etc.), which actually happens quite often. This complicates the model somewhat, nevertheless, it is possible to obtain an optimal solution using the same methods that are used to solve the main problem and do not consider the temporary shutdown of production. Naturally, when additional conditions are introduced, some difficulties appear, which are discussed in the article, it also shows the way to overcome them. The main method of solving such problems involves knowing all the parameters of the processed batches, including those that cannot be known a priori (one can only know empirical estimates of these coefficients). Based on two main special cases, examples of absolute stability, conditional stability in the period J and local stability in the period J of optimal schedules are given. Their benefits for production are discussed. Due to the specifics of the objective function, minor deviations of its parameters have little effect on its value. In the article, optimal schedules are found for two main special cases, their absolute stability is proved, with the proximity of the initial parameters to such special cases, quasi-optimal processing schedules can be built on their basis, and due to absolute stability, they remain quasi-optimal schedules, considering the shutdown.

The problem of availability, sufficiency, and accessibility of food for the population of the country is one of the factors determining the sovereignty of the country and can be solved only by their joint consideration and optimization of both cultivation and processing of products [25]. Sugar in Russia is of great strategic importance, and the sugar industry has a significant structure-forming influence on the establishment of sectoral proportions in the economy, therefore, optimization of the sugar beet processing schedules is important, especially considering that it practically does not require significant costs for its introduction into production.

Acknowledgement

The article was carried out under the contract No SSZ-1771 dated 22.04.2021. on the implementation of R&D on the topic: "Creation of high-tech sugar production on the basis of JSC "Sergach Sugar Plant", within the framework of the Agreement on the provision of subsidies from the federal budget for the development of cooperation between the Russian educational organization of higher education and the organization of the real sector of the economy in order to implement a comprehensive project to create high-tech production No. 075-11-2021-038 of 24.06.2021. (IGC 000000S407521QLA0002).

References

1. L.M. Nevdyayev, Telecommunication technologies. English-Russian explanatory dictionary-handbook (Moscow: ICNTI, 2002) 582.
2. D.G. Rodionov, D.A. Alferyev, Scientific and technical bulletin of SPbSPU. Economic sciences, **13(5)**, 106–119 (2020) <https://doi.org/10.18721/JE.13508>

3. A.S. Eliseev, M.B. Gitman, S.S. Sukhantsev, UBS **42**, 257-272 (2013)
4. V.I. Sviridov, N.N. Petrenko, International Agricultural Journal **4**, 28-30 (2005).
5. M.G. Zaitsev, Management optimization methods for managers. Computer-oriented approach (M.: Delo, 2016) 116.
6. O.S. Bychkov, D.Y. Khusainov, O.I. Liashenko [et al.], Journal of advanced research in law and economics **10(8(46))**, 2282-2296 (2019) [https://doi.org/10.14505/jarle.v10.8\(46\).07](https://doi.org/10.14505/jarle.v10.8(46).07).
7. P. Dvurechensky, A. Gasnikov, S. Omelchenko, A. Tiurin, A Stable, Lecture Notes in Computer Science, **12095 LNCS**, 406-423 (2020) https://doi.org/10.1007/978-3-030-49988-4_28.
8. D.E. Orlova, L.V. Rossikhina, Bulletin of the Voronezh Institute of High Technologies **1(36)**, 95-99 (2021)
9. F.K. Mutasa, S.D. Hove-Musekwa, B. Jones [et al.] Discrete dynamics in nature and society, **2022**, 8504993 (2022) <https://doi.org/10.1155/2022/8504993>
10. A.F. Izmailov, Sensitivity in optimization (M.: Fizmatlit, 2006) 248
11. S.N. Medvedev, Bulletin of Voronezh Inst. of FPSR of Russia **1**, 37-41 (2011)
12. S.N. Medvedev, Finding the stability domain of the interval assignment problem, *Bulletin of Voronezh State Technical University* **7(10)**, 51-54 (2011)
13. A.Ya. Asnina, N.B. Baeva, G.D. Chernyshova, Computational methods of linear optimization (Voronezh: Voronezh State University, 1987) 156
14. L.I. Lopatnikov, Economic and mathematical dictionary (Moscow: Delo, 2003) 520.
15. B. Bandi, Fundamentals of linear programming (M.: Radio and Communications, 1989) 176
16. H.W. Kuhn, Nav. Res. Logistics Quart. **2**, 83-97 (1955)
17. D.P. Bertsekas, *Auction algorithms for network flow problems: a tutorial introduction*, Computational optimization 7-66 (1992)
18. E.V. Alekseeva, Construction of mathematical models of integer linear programming. Examples and tasks: Textbook (Novosibirsk: Novosibirsk State University, 2012) 131
19. D.V. Balandin, [and etc.], Communications in Computer and Information Science **1750**, 227-238 (2022) https://doi.org/10.1007/978-3-031-24145-1_19.
20. D.V. Balandin, [and etc.], Supercomputing. Lecture Notes in Computer Science, book series LNCS, **13708** 409-422 (2022) https://doi.org/10.1007/978-3-031-22941-1_30.
21. D.V. Balandin [et al.] The strategy of processing sugar beet batches with parameters close to its withering. The second All-Russian Scientific and Practical Seminar "Mathematical and computer modeling and business analysis in the conditions of digitalization of the economy". Nizhny Novgorod, April 22, 2022 Collection of works. (Nizhny Novgorod: UNN, 2022) 10-18.
22. D.V. Balandin, O.A. Kuzenkov, A.I. Egamov, Eurasian Union of Scientists. Series: Technical, Physical and Mathematical Sciences **9(102)**, 33-39 (2022) <https://doi.org/10.31618/ESU.2413-9335.2022.1.102.1724>
23. V.N. Kukhar, A.P. Chernyavsky, L.I. Chernyavskaya, Yu.A. Mokanyuk, Sugar **1**, 18-36 (2019)
24. L.V. Radzivilovsky, Generalization of permutation inequality and Mongolian inequality (M.: Publishing House of the ICNMO. – Matem. ed. series 3 10 210-224, 2006)
25. D.V. Balandin, Yu.A. Kuznetsov, Economic analysis: theory and practice **20(11(518))**, 2134-2150 (2021)