Alternative methods for solving a hyperbolic system of equations in a simplified water hammer problem

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Abstract. The paper addresses a mathematical model of the transient process in a pipeline, namely, a water hammer model. Based on this model, a mixed problem for a system of partial differential equations of hyperbolic type is set. In this paper, we obtain formulas for the exact solution of the Cauchy problem for a linear homogeneous hyperbolic system. We compare the effectiveness of explicit and implicit numerical methods for solving a simplified system of water hammer equations.

1 INTRODUCTION

Pressure pipeline and hydraulic systems (PLS) of various types (heat, water supply, technological purposes, hydraulic drives of test benches in aviation and mechanical engineering, hydraulic structures, thermal power plants, etc.) are widely used and have an important position in energy, water management, industry, housing and communal services and other sectors of the economy. In this regard, the topic of the PLS study is very relevant. The reliability and safety of the functioning the PLS crucially depends on the competent organization of warning systems and protection against water hammer, which is impossible without the using effective methods for modeling such processes.

Despite the large number of works devoted to the study of unsteady fluid flows in PLS [1-5, etc.], the present day practice of their operation set new problems related to the widespread automation of PLS, increasingly active consumer behavior, the introduction of mathematical modeling methods into the practice of operational control and management of variable modes. At the same time, in addition to the traditional requirements for the adequacy and accuracy of the calculation results, the problems of construction of high-speed computational methods for calculating non-stationary modes of large-dimensional PLS are increasingly coming to the fore.

In this paper, we consider the problem of modeling a transient process in a pipeline caused by a pressure sudden change (water hammer problem). The model of such processes is described by partial differential equations of hyperbolic type. Traditionally, the water

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hammer problems are solved numerically, using various modifications of the characteristics method of and the running count scheme [1-7]. A detailed review of such methods is given in [1]. Authors of [1, 2] recommend the using explicit-implicit schemes for the water hammer problem.

The purpose of this work is to study approaches to the development of analytical and effective numerical methods for solving water hammer problem, which will reduce the duration of machine calculations. It is not possible to obtain an analytical solution to the classical water hammer problem, but such a solution is possible to obtain with a simplification of the model. In this paper, formulas for the exact solution to the Cauchy problem for a linear homogeneous hyperbolic system are obtained. The authors also follow a methodological goal – to obtain an analytical solution of simplified problem and compare the effectiveness of four numerical methods, including the traditional characteristics method.

2. STATEMENT OF THE WATER HAMMER PROBLEM

Fluid motion occurs according to fundamental physical laws: the law of conservation of mass, the law of conservation of momentum and the law of conservation of energy [1-3, 5, 8]. If the fluid motion is continuous in a separate volume, then the conservation law in this volume can be written in the form of differential equations.

An incompressible Newtonian fluid (for example, water) in a pressure horizontal pipeline is considered. The mathematical model of water hammer in a pipeline is described by a system of partial differential equations of hyperbolic type [1-5, 8]. The model includes the equation of motion

$$\frac{\partial p}{\partial l} + \alpha_0 \frac{\partial x}{\partial t} + \alpha_1 |x| x = 0 \qquad (1)$$

and the continuity equation

$$\frac{\partial x}{\partial l} + \alpha_2 \frac{\partial p}{\partial t} = 0, \qquad (2)$$

where p is the pressure, x is the mass flow rate, and the coefficients

$$\alpha_0 = \frac{4}{\pi D^2}, \quad \alpha_1 = \frac{8\lambda}{\pi^2 D^5 \rho}, \quad \alpha_2 = \frac{\pi D^2}{4a^2} \qquad (3)$$

are constant values. Here D is the inner diameter of the pipeline; λ is coefficient of friction resistance; ρ is density of the transported fluid; a is velocity of propagation of pressure waves. The time t and the coordinate of the pipe length l are independent variables.

In this formulation thermal processes and convective terms are not taken into account.

Let us set the initial conditions. The first condition takes into account the friction pressure loss:

$$p(l,t_0) = p_0 - \alpha_1 x_0^2 l, \quad 0 \le l \le L, \qquad (4)$$

the second one specifies a constant flow rate along the length of the pipeline at a time t_0:

$$x(l,t_0) = x_0, \quad 0 \le l \le L. \qquad (5)$$

Additionally, conditions are set at the borders. An external constant pressure source operates at the left end of the pipeline:

$$p(l,t)\Big|_{l=0} = p_0, \quad t_0 \le t \le T. \qquad (6)$$

At the right end of the pipeline, the fluid stream begins to close at the moment t_0 (for example, according to the linear law). After that, the fluid stream is completely blocked for a time period T_1:

$$x(l,t)\ \Big|_{l=L} = \begin{cases} x_0\left(1 - t/T_1\right), & t_0 \le t < T_1; \\ 0, & T_1 \le t \le T. \end{cases} \tag{7}$$

As a result, a hydraulic transient occurs in the pipeline, which is called a water hammer. It is required to find functions $p(l,t)$ and $x(l,t)$ in the area of changes in independent variables $G = [0, L] \times [t_0, T]$, where L is the length of the pipeline. The discriminant of the system (1), (2) is positive, so we are dealing with a hyperbolic type equation. Thus, we have a mixed problem (1)-(7). Note that there is a general solution to the system (1), (2) (see [9], p. 43). The monograph [5] cosiders the existence of a solution to the Cauchy problem (p. 91), a mixed problem (p. 94) for systems of nonlinear hyperbolic equations. Autors of [10] propose an exact solution of the system in the presence of an inhomogeneous equation of fluid motion in it.

2.1 Simplified water hammer equations

As a test we will use a problem that contains specific difficulties of this class of problems and whose exact solution is known. Consider the simplified form of equation (1), excluding the term of friction losses $\alpha_1 \,|\, x \,|\, x$, then the system (1), (2) will take the form

$$\frac{\partial p}{\partial l} + \alpha_0 \frac{\partial x}{\partial t} = 0, \tag{8}$$

$$\frac{\partial x}{\partial l} + \alpha_2 \frac{\partial p}{\partial t} = 0. \tag{9}$$

The solution to the Cauchy problem for the simplified system (8), (9) can be obtained analytically [9].

3 EXACT SOLUTION TO THE CAUCHY PROBLEM FOR A LINEAR HOMOGENEOUS HYPERBOLIC SYSTEM

Statement 1. *Let the functions and be continuous together with their derivatives* $\forall (l,t)$ *such that* $l \in [0, L]$, $L \in \mathbb{R}_+$, $0 \le t \le T$, $T \in \mathbb{R}_+$. *Then the Cauchy problem for system (8), (9) with conditions*

$$x(l,0) = \varphi(l), \quad p(l,0) = \Psi(l) \tag{10}$$

has a solution of the form

$$x(l,t) = \frac{\varphi(l - at) + \varphi(l + at)}{2} + \frac{\alpha_2 a(\Psi(l - at) - \Psi(l + at))}{2}, \tag{11}$$

$$p(l,t) = \frac{\Psi(l - at) + \Psi(l + at)}{2} + \frac{\alpha_0 a(\varphi(l - at) - \varphi(l + at))}{2}, \tag{12}$$

where

$$a = 1/\sqrt{\alpha_0 \alpha_2}. \tag{13}$$

One can verify the validity of the statement by directly substituting functions of the form (11) and (12) (taking into account the designation (13)) into equations (8), (9) and conditions (10).

The solution (11), (12) exists and is unique in the domain of the determinancy of the segment $[0, L]$. This domain is enclosed between the segment itself, the left characteristic $l + at = c_1$ of the system (8), (9) drawn through the right end of the segment, and the right characteristic $l - at = c_2$ drawn through the left end of the segment. This is the triangle OAL in Fig. 1. In Section 4.1, we will compare the numerical solution of the problem (8)-(10) with the solution (11), (12) on the domain of the determinancy of the segment.

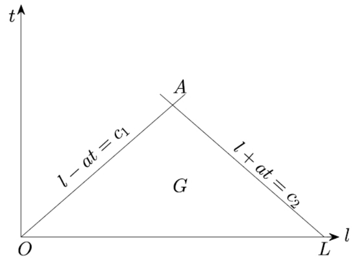

Fig.1. Domain of the definiteness of the segment

4 On the numerical solution to hyperbolic type equations

For hyperbolic type problems, difference methods are often used because they are universal and theoretically well developed [5-7]. The difference solution is built on a grid, which is introduced in the area of variable changes $G(l, t)$. All derivatives that are included in the equations and boundary conditions are replaced by differences in the values of the functions at the nodes of the grid. The resulting algebraic equations are a difference scheme. An approximate (difference) solution is obtained by solving the resulting algebraic system.

Running counting schemes are designed to solve mixed problems. They are the simplest and allow numerically solving even very complex problems with good accuracy with a moderate amount of calculations [6]. Let us choose three templates shown in Fig. 2.

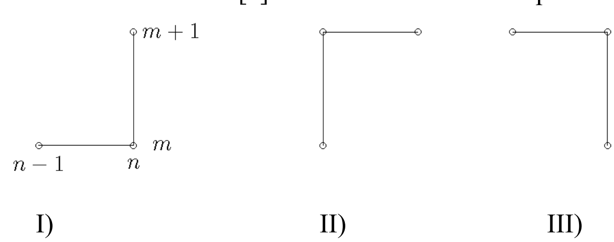

Fig. 2. Three-point patterns of running account schemes

If the solution $p(l,t)$, $x(l,t)$ and its second derivatives are continuous, then the numerical solution obtained by schemes I-III converges with the first order of accuracy (this follows from the convergence theorems [5, 6]). Implicit schemes II and III converge unconditionally. Scheme I is stable on the right side in $\|\,.\,\|_C$ when the Courant condition

$$a\tau \le h, \tag{14}$$

is met, where the coefficient a is determined by the formula (13).

The method of characteristics is one of the most common methods for integrating systems of hyperbolic equations in water hammer problems [2, 4, 5, 11]. The method is constructed under the assumption that the initial system has a smooth solution in the domain G and the problem of finding a solution in the neighborhood of characteristics is considered. Along the characteristics, partial differential equations turn into ordinary differential equations. The solution to these equations is obtained and proceeds to the solution of the original system of partial differential equations.

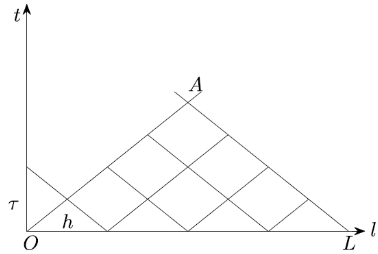

Fig. 3. Grid of the characteristics method

The numerical solution of the characteristic relations is built on a grid (Fig. 3), which takes into account the characteristic directions and is called characteristic. In our case, the Euler method was used for the numerical solution of ordinary differential equations.

4.1 Results of numerical calculations

Example. We will set the parameters of the pipeline $D = 0,4$ (m), $L = 40$ (m), $a = 1000$ (m/s), $\rho = 1000$ (kg/m3), roughness coefficient $k = 0,5$ (mm), coefficient $\lambda = 0,11\sqrt[4]{k/D}$ (in formula (3)), initial values $x_0 = 500$ (kg/s), $p_0 = 980665$ (Pa). The system (8), (9) with initial conditions $p(l,0) = -\alpha_0 p_0 l$, $x(l,0) = x_0 + l$, has a solution $p(l,t) = -\alpha_0 p_0 l - \frac{1}{\alpha_2} t$, $x(l,t) = x_0 + l + p_0 t$. Table 1 shows the error values $\varepsilon^{h,\tau} = \max\limits_{\Delta} | x(l_n, t_m) - x_n^m |$ of the numerical solution for the flow rate. Here Δ is the domain of determinancy of the segment $[0, L]$. The symbol «*» means that the error has exceeded the specified level (10^2). In the first two rows of the table, the values of the grid steps satisfy condition (14). The table shows that explicit methods (I and the method of characteristics) are effective for small steps, but not effective for large time steps. Implicit methods (II, III) are the opposite.

Table 1. The error of the numerical solution

steps of the numerical scheme		method			
τ	h	I	II	III	characteristics method
0.005	5	$9.10 \cdot 10^{-12}$	$2.18 \cdot 10^{-11}$	*	$2.50 \cdot 10^{0}$
0.0025	2.5	$3.23 \cdot 10^{-10}$	$9.24 \cdot 10^{-9}$	*	$1.25 \cdot 10^{0}$
0.01	5	$3.64 \cdot 10^{-11}$	$2.28 \cdot 10^{-12}$	$2.58 \cdot 10^{-10}$	*
0.05	5	$2.21 \cdot 10^{-3}$	$5.82 \cdot 10^{-11}$	$1.75 \cdot 10^{-10}$	*

0.1	5	$1.40\cdot10^{-7}$	$1.16\cdot10^{-10}$	$2.33\cdot10^{-10}$	*
1	5	$1.70\cdot10^{-5}$	$9.31\cdot10^{-10}$	$1.86\cdot10^{-9}$	*
10	5	$4.47\cdot10^{-2}$	$7.45\cdot10^{-9}$	$2.98\cdot10^{-8}$	*

5 Conclusion

We obtain an exact solution to the Cauchy problem for a linear homogeneous hyperbolic system describing a simplified water hammer problem. The results of the numerical solution of the test case constructed for the same case are compared. Calculations were performed using an explicit running account scheme, an explicit method of characteristics and two implicit running account schemes. The results of the calculations showed that the use of implicit methods makes it possible to increase the numerical time step without loss of accuracy, while reducing the duration of calculations.

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