Mathematical modeling of salt concentration change process in two-layer aqueous media

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Abstract. A mathematical model was developed to monitor and predict changes in salt concentration in groundwater in a two-layer environment, representing the process of salinization and swamping. A qualitative analysis of scientific research on the problem is provided. Since the process is represented by a differential equation, effective numerical computational algorithms were developed to solve the problem of salt concentration change in groundwater based on a stable disclosure scheme with high-order accuracy relative to time and spatial variables. Mathematical models of salt transport during geo-filtration processes were improved, and computational algorithms were developed, taking into account such important parameters as the physical and mechanical properties: water loss coefficient, diffusion coefficients, and filtration rates depending on the water level. The results obtained based on these developed numerical algorithms were analyzed. Before introducing new technologies, recommendations were developed to predict changes in salt content in water using an effective numerical algorithm. Based on the laws of motion of wastewater flows, it was noted that it is possible to conduct experiments using algorithms to determine the changes in distance and velocity of wastewater propagation with water-soluble chemicals and active properties in soil layers.

1 Introduction

Currently, targeted research is being conducted worldwide to study groundwater flows and geofiltration processes. The main issues in this direction are the development of improved mathematical models of the change in groundwater level and mineralization processes in one- and two-layer environments and effective numerical computational algorithms and software tools. Therefore, the development and improvement of mathematical models of groundwater level changes and mineralization processes, as well as effective computational algorithms to solve these problems, are the most pressing scientific problems. Great attention is paid to developing mathematical models and numerical computational algorithms to study complex hydrodynamic and geofiltration processes in developed foreign countries, including Japan, Denmark, Canada, Russia, China, Germany, and India.

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Given the acute shortage of water resources, especially in Central Asia, the water supply problem to the population in the environmentally unfavorable areas of Khiva, Bukhara, and Karakalpakstan is urgent. In such conditions, one of the main sources of water supply to the population is groundwater. Reducing energy and labor costs, increasing crop efficiency while meeting environmental requirements, and improving the quality of products are the key issues in the development of the agricultural sector. This, in turn, is related to the intensity of water reclamation in agricultural landscapes, agricultural drainage calculation, and land irrigation.

In particular, 45% of irrigated lands in the Republic of Uzbekistan (approximately two million hectares) are prone to salinization and or are saline soils. Evaporation is one of the main reasons for this phenomenon. In particular, as of October 1, 2018, according to the Regional Melioration Expedition under the "Amu-Bukhara" Irrigation Basin Department, 85.8% of 274,612 hectares of irrigated land in Bukhara region is saline to varying degrees, including 61.7% of weak, 21.7% of average, 2.4% of strongly saline soils[1, 2].

It should be noted that in Central Asia, the Caucasus, and other regions, irrigation of agricultural lands based on groundwater through drainage reaches 30% [3]. The process of soil salinization occurs due to the rise in the groundwater level, resulting in a sharp decline in soil fertility. This process causes enormous damage to the environment and poses risks to human life and health.

In the western part of the country (the lower reaches of the Zarafshan and the Kashkadarya, Syrdarya, Amudarya Rivers, and Central Kyzylkum basins), groundwater is highly mineralized and hard. Over the last 10-15 years, the water of groundwater lenses formed along large rivers (Amudarya and irrigation canals) used for freshwater supply to the Khorezm region and the Republic of Karakalpakstan; the requirements of national standards due to increased mineralization and hardness (irrigation effect). In such cases, an urgent task from a practical point of view in forming freshwater reserves is to study the process of changing the salt concentration in the groundwater layer [4].

To study the process of changing the salt concentration in the aquifer, it is necessary to develop an improved mathematical model that describes the main features of the object. In [5], M.S.Troyansky presented the process of groundwater filtration and described the types and methods of process modeling. Particular attention was paid to the numerical modeling of groundwater filtration. Problems with filtration flow based on the theory of geofiltration for the steady state of fluid according to Darcy's law were considered in [6, 7]. The results of numerical calculations were analyzed based on hydrodynamic laws. N. Ravshanov in [8] developed numerical modeling of soil moisture content and salt migration processes and a numerical algorithm based on the Samarskii-Fryazinov vector scheme. Models were proposed to obtain reliable data on changes in groundwater levels, optimize the calculation of farming drainage and adjust the water regime management of agricultural lands [9].

Anderson in [10] considered the problem of the two-dimensional steady-state flow of groundwater. An analytical solution was developed to study the process of water interaction with the surface of groundwater. A modeling system based on soil moisture content and groundwater dynamics was considered by Xi Chen for models of hydrological processes [11]. A mathematical model for the numerical solution of non-stationary problems of two-component fluid flow in a porous medium was proposed by E.B. Soboleva [12]. An appropriate approach to the Soil Conservation Service model and visual Modular Three-dimensional Finite-difference Groundwater Flow Model to assess the impact of soil and water conservation and groundwater is based on the water balance principle [13]. H. Mutao and T. Yong in [14] conducted a physical analysis of freshwater resources considering climatic changes. A physical model was proposed to predict and monitor groundwater and pressure water levels changes.

A mathematical model of the salinization process was developed by A. Vlasyuk [15], considering infiltration in water-saturated stratified soils. In [16], R. Rong proposed a mathematical model of groundwater variation considering the temperature gradient in the northern coastal regions of China. The results of the numerical solution to the problem show that the soil temperature gradient has a substantial effect. Tenalem Ayenew in [17] developed a stationary groundwater model to analyze the groundwater hydrodynamics in the Akaki catchment basin in Addis Ababa. The issues of water supply to the city were considered. The developed model was used in the dynamics of groundwater flow to determine the interaction of groundwater and surface water and to study the effect of well pumping.

Based on the above, the analysis of the scientific literature on mathematical modeling of geofiltration processes to study the groundwater flow has shown that several important theoretical and practical results were obtained in this field.

2 Materials and Methods

The research shows that the changes in the mineralization level of groundwater and pressure water require the development of more elaborate methods of studying the issues related to the assessment of land reclamation. Predicting changes in groundwater levels and mineralization is possible by developing mathematical models that describe the process under study. In developing a mathematical model representing the interaction of groundwater and pressure water, a system of linear and nonlinear differential equations with basic laws of hydrodynamics and appropriate boundary conditions was used. A mathematical model that accurately reflects this process was developed to monitor and predict changes in groundwater salt content. The following initial and boundary conditions were used [18, 19]:

$$\left.\begin{aligned}
\mu_1 h\frac{\partial\theta_1}{\partial t} &= \frac{\partial}{\partial x}\left(D_1 h\frac{\partial\theta_1}{\partial x}\right)+\frac{\partial}{\partial z}\left(D_1 h\frac{\partial\theta_1}{\partial z}\right)-v_x h\frac{\partial\theta_1}{\partial x}-v_z h\frac{\partial\theta_1}{\partial z}+f\theta_f,\\
\mu_2 H\frac{\partial\theta_2}{\partial t} &= \frac{\partial}{\partial x}\left(D_2 H\frac{\partial\theta_2}{\partial x}\right)+\frac{\partial}{\partial z}\left(D_2 H\frac{\partial\theta_2}{\partial z}\right)-v_x H\frac{\partial\theta_2}{\partial x}-v_z H\frac{\partial\theta_2}{\partial z}+f_1\theta_{1f},
\end{aligned}\right\} \tag{1}$$

Here $\theta_1(x,z,t)$, $\theta_2(x,z,t)$ are the salt concentrations in the subsurface and pressure aquifers; $h(x,z,t)$, $H(x,z,t)$ are the levels of groundwater and pressure water; μ is water loss coefficient; k is filtration coefficient; D_1, D_2 are diffusion coefficients; θ_{1f}, θ_{2f} are salt concentrations (in infiltration water): v_x, v_z are filtration rate components.

System (1) is solved with the following initial and boundary conditions:

$$(\theta_1)\big|_{t=t_0}=(\theta_1)_0,\ \ (\theta_2)\big|_{t=t_0}=(\theta_2)_0, \tag{2}$$

$$\mu_1 h\frac{\partial\theta_1}{\partial x}\bigg|_{x=0}=-(\theta_1-(\theta_1)_0),\ \ \mu_1 h\frac{\partial\theta_1}{\partial x}\bigg|_{x=L_x}=(\theta_1-(\theta_1)_0), \tag{3}$$

$$\mu_1 h\frac{\partial\theta_1}{\partial z}\bigg|_{z=0}=-(\theta_1-(\theta_1)_0),\ \ \mu_1 h\frac{\partial\theta_1}{\partial z}\bigg|_{z=L_z}=(\theta_1-(\theta_1)_0), \tag{4}$$

$$\mu_2 H\frac{\partial\theta_2}{\partial x}\bigg|_{x=0}=-(\theta_2-(\theta_2)_0),\ \ \mu_2 H\frac{\partial\theta_2}{\partial x}\bigg|_{x=L_x}=(\theta_2-(\theta_2)_0), \tag{5}$$

$$\mu_2 H\frac{\partial\theta_2}{\partial z}\bigg|_{z=0}=-(\theta_2-(\theta_2)_0),\ \ \mu_2 H\frac{\partial\theta_2}{\partial z}\bigg|_{z=L_z}=(\theta_2-(\theta_2)_0), \tag{6}$$

$$\theta_1\big|_{z=m-0} = \theta_2\big|_{z=m+0}, \tag{7}$$

$$D_1 h\frac{\partial\theta_1}{\partial y}\bigg|_{z=m-0} = D_2 H\frac{\partial\theta_2}{\partial z}\bigg|_{z=m+0}. \tag{8}$$

$(\theta_1)_0$, $(\theta_2)_0$ are the initial values of salt concentrations in the subsurface and pressure aquifers; m is the thickness; t_0 isinitial time.

To solve problems (1) - (8), we introduce the following dimensionless quantities:

$$h^* = \frac{h}{h_0},\ H^* = \frac{H}{H_0},\ \ x^* = \frac{x}{L_x},\ \ z^* = \frac{z}{L_z},\ \tau = \frac{m_0}{\mu_1 n_0 L_x^2}t,\ m^* = \frac{m}{m_0},\ D_1^* = \frac{D_1}{(D_1)_0},\ D_2^* = \frac{D_2}{(D_2)_0},$$
$$\theta_1^* = \frac{\theta_1}{(\theta_1)_0},\ \theta_2^* = \frac{\theta_2}{(\theta_2)_0}.$$

In this case, problems (1) - (8) take the following form:

$$\left.\begin{array}{l} h^*\dfrac{\partial\theta_1^*}{\partial\tau} = \dfrac{(D_1)_0 n_0}{m_0}\dfrac{\partial}{\partial x^*}\left(D_1^* h^*\dfrac{\partial\theta_1^*}{\partial x^*}\right) + \dfrac{(D_1)_0 n_0 L_x^2}{(k_1)_0 m_0 L_z^2}\dfrac{\partial}{\partial z^*}\left(D_1^* h^*\dfrac{\partial\theta_1^*}{\partial z^*}\right) - \dfrac{n_0 L_x}{m_0}v_x h^*\dfrac{\partial\theta_1^*}{\partial x^*} - \\ \qquad - \dfrac{n_0 L_x^2}{m_0 L_z^2}v_z h^*\dfrac{\partial\theta_1^*}{\partial z^*} + \dfrac{n_0 L_x^2}{m_0 h_0(\theta_1)_0}f\cdot\theta_f, \\ H^*\dfrac{\partial\theta_2^*}{\partial\tau} = \dfrac{\mu_1 n_0 (D_2)_0}{\mu_2 m_0}\dfrac{\partial}{\partial x^*}\left(D_2^* H^*\dfrac{\partial\theta_2^*}{\partial x^*}\right) + \dfrac{\mu_1 n_0 (D_2)_0 L_x^2}{\mu_2 m_0 L_z^2}\dfrac{\partial}{\partial z^*}\left(D_2^* H^*\dfrac{\partial\theta_2^*}{\partial z^*}\right) - \\ \quad - \dfrac{\mu_1 n_0 (D_2)_0 L_x}{\mu_2 m_0}v_x H^*\dfrac{\partial\theta_2^*}{\partial x^*} - - \dfrac{\mu_1 n_0 L_x^2}{\mu_2 m_0 L_z}v_z H^*\dfrac{\partial\theta_2^*}{\partial z^*} + \dfrac{\mu_1 n_0 L_x^2}{\mu_2 m_0 H_0(\theta_2)_0}f_1\cdot\theta_{1f}, \end{array}\right\} \tag{9}$$

And boundary conditions are:

$$\frac{\mu_1 h_0(\theta_1)_0}{L_x}h^*\frac{\partial\theta_1^*}{\partial x^*}\bigg|_{x^*=0} = -((\theta_1)_0\theta_1^* - (\theta_1)_0),\ \ \frac{\mu_1 h_0(\theta_1)_0}{L_x}h^*\frac{\partial\theta_1^*}{\partial x^*}\bigg|_{x^*=1} = ((\theta_1)_0\theta_1^* - (\theta_1)_0), \tag{10}$$

$$\frac{\mu_1 h_0(\theta_1)_0}{L_z}h^*\frac{\partial\theta_1^*}{\partial z^*}\bigg|_{z^*=0} = -((\theta_1)_0\theta_1^* - (\theta_1)_0),\ \ \frac{\mu_1 h_0(\theta_1)_0}{L_z}h^*\frac{\partial\theta_1^*}{\partial z^*}\bigg|_{z^*=1} = ((\theta_1)_0\theta_1^* - (\theta_1)_0), \tag{11}$$

$$\frac{\mu_2 H_0(\theta_2)_0}{L_x}H^*\frac{\partial\theta_2^*}{\partial x^*}\bigg|_{x^*=0} = -((\theta_2)_0\theta_2^* - (\theta_2)_0),\ \ \frac{\mu_2 H_0(\theta_2)_0}{L_x}H^*\frac{\partial\theta_2^*}{\partial x^*}\bigg|_{x^*=1} = ((\theta_2)_0\theta_2^* - (\theta_2)_0), \tag{12}$$

$$\frac{\mu_2 H_0(\theta_2)_0}{L_z}H^*\frac{\partial\theta_2^*}{\partial z^*}\bigg|_{z^*=0} = -((\theta_2)_0\theta_2^* - (\theta_2)_0),\ \ \frac{\mu_2 H_0(\theta_2)_0}{L_z}H^*\frac{\partial\theta_2^*}{\partial z^*}\bigg|_{z^*=1} = ((\theta_2)_0\theta_2^* - (\theta_2)_0), \tag{13}$$

$$(\theta_1)_0\theta_1^*\bigg|_{z^*=\frac{m-0}{L_z}} = (\theta_2)_0\theta_2^*\bigg|_{z^*=\frac{m+0}{L_z}}, \tag{14}$$

$$\frac{(D_1)_0 h_0(\theta_1)_0}{L_z}D_1^* h^*\frac{\partial\theta_1^*}{\partial y^*}\bigg|_{y^*=\frac{m-0}{L_z}} = \frac{(D_2)_0 H_0(\theta_2)_0}{L_z}D_2^* H^*\frac{\partial\theta_2^*}{\partial z^*}\bigg|_{z^*=\frac{m+0}{L_z}}.. \tag{15}$$

To simplify the following calculation sequences, we omit sign "*" in equations (9) - (15) and write the equations as:

$$\left.\begin{aligned} h\frac{\partial\theta_1}{\partial\tau} = \frac{(D_1)_0 n_0}{m_0}\frac{\partial}{\partial x}\left(D_1 h\frac{\partial\theta_1}{\partial x}\right) + \frac{(D_1)_0 n_0 L_x^2}{m_0 L_z^2}\frac{\partial}{\partial z}\left(D_1 h\frac{\partial\theta_1}{\partial z}\right) - \frac{n_0 L_x}{m_0} v_x h\frac{\partial\theta_1}{\partial x} - \\ - \frac{n_0 L_x^2}{m_0 L_z^2} v_z h\frac{\partial\theta_1}{\partial z} + \frac{n_0 L_x^2}{m_0 h_0 (\theta_1)_0} f\theta_f, \\ H\frac{\partial\theta_2}{\partial\tau} = \frac{\mu_1 n_0 (D_2)_0}{\mu_2 m_0}\frac{\partial}{\partial x}\left(D_2 H\frac{\partial\theta_2}{\partial x}\right) + \frac{\mu_1 n_0 (D_2)_0 L_x^2}{\mu_2 m_0 L_z^2}\frac{\partial}{\partial z}\left(D_2 H\frac{\partial\theta_2}{\partial z}\right) - \\ - \frac{\mu_1 n_0 (D_2)_0 L_x}{\mu_2 m_0} v_x H\frac{\partial\theta_2}{\partial x} - \frac{\mu_1 n_0 L_x^2}{\mu_2 m_0 L_z} v_z H\frac{\partial\theta_2}{\partial z} + \frac{\mu_1 n_0 L_x^2}{\mu_2 m_0 H_0 (\theta_2)_0} f_1\theta_{1f}. \end{aligned}\right\} \quad (16)$$

entering the following definitions to (16), the problem is simplified

$$\varphi = \frac{(D_1)_0 n_0}{m_0}, \quad \varphi_1 = \frac{(D_1)_0 n_0 L_x^2}{m_0 L_z^2}, \quad \varphi_2 = \frac{n_0 L_x}{m_0}, \quad \varphi_3 = \frac{n_0 L_x^2}{m_0 L_z^2}, \quad \varphi_4 = \frac{n_0 L_x^2}{m_0 h_0 (\theta_1)_0}, \quad \psi = \frac{\mu_1 n_0 (D_2)_0}{\mu_2 m_0},$$

$$\psi_1 = \frac{\mu_1 n_0 (D_2)_0 L_x^2}{\mu_2 m_0 L_z^2}, \quad \psi_2 = \frac{\mu_1 n_0 (D_2)_0 L_x}{\mu_2 m_0}, \quad \psi_3 = \frac{\mu_1 n_0 L_x^2}{\mu_2 m_0 L_z}, \quad \psi_4 = \frac{\mu_1 n_0 L_x^2}{\mu_2 m_0 H_0 (\theta_2)_0}, \quad D_1 = D_1 h, \quad D_2 = D_2 H$$

we express as

$$\left.\begin{aligned} h\frac{\partial\theta_1}{\partial\tau} = \varphi\frac{\partial}{\partial x}\left(D_1\frac{\partial\theta_1}{\partial x}\right) + \varphi_1\frac{\partial}{\partial z}\left(D_1\frac{\partial\theta_1}{\partial z}\right) - \varphi_2 v_x h\frac{\partial\theta_1}{\partial x} - \varphi_3 v_z h\frac{\partial\theta_1}{\partial z} + \varphi_4 f\theta_f, \\ H\frac{\partial\theta_2}{\partial\tau} = \psi\frac{\partial}{\partial x}\left(D_2\frac{\partial\theta_2}{\partial x}\right) + \psi_1\frac{\partial}{\partial z}\left(D_2\frac{\partial\theta_2}{\partial z}\right) - \psi_2 v_x H\frac{\partial\theta_2}{\partial x} - \psi_3 v_z H\frac{\partial\theta_2}{\partial z} + \psi_4 f_1\theta_{1f}, \end{aligned}\right\} \quad (16^*)$$

Dimensional forms of boundary conditions can be expressed by the following equations:

$$\frac{\mu_1 h_0(\theta_1)_0}{L_x} h\frac{\partial\theta_1}{\partial x}\bigg|_{x=0} = -((\theta_1)_0\theta_1 - (\theta_1)_0), \quad \frac{\mu_1 h_0(\theta_1)_0}{L_x} h\frac{\partial\theta_1}{\partial x}\bigg|_{x=1} = ((\theta_1)_0\theta_1 - (\theta_1)_0), \quad (17)$$

$$\frac{\mu_1 h_0(\theta_1)_0}{L_z} h\frac{\partial\theta_1}{\partial z}\bigg|_{z=0} = -((\theta_1)_0\theta_1 - (\theta_1)_0), \quad \frac{\mu_1 h_0(\theta_1)_0}{L_z} h\frac{\partial\theta_1}{\partial z}\bigg|_{z=1} = ((\theta_1)_0\theta_1 - (\theta_1)_0), \quad (18)$$

$$\frac{\mu_2 H_0(\theta_2)_0}{L_x} H\frac{\partial\theta_2}{\partial x}\bigg|_{x=0} = -((\theta_2)_0\theta_2 - (\theta_2)_0), \quad \frac{\mu_2 H_0(\theta_2)_0}{L_x} H\frac{\partial\theta_2}{\partial x}\bigg|_{x=1} = ((\theta_2)_0\theta_2 - (\theta_2)_0), \quad (19)$$

$$\frac{\mu_2 H_0(\theta_2)_0}{L_z} H\frac{\partial\theta_2}{\partial z}\bigg|_{z=0} = -((\theta_2)_0\theta_2 - (\theta_2)_0), \quad \frac{\mu_2 H_0(\theta_2)_0}{L_z} H\frac{\partial\theta_2}{\partial z}\bigg|_{z=1} = ((\theta_2)_0\theta_2 - (\theta_2)_0), \quad (20)$$

$$(\theta_1)_0\theta_1\bigg|_{z=\frac{m-0}{L_z}} = (\theta_2)_0\theta_2\bigg|_{z=\frac{m+0}{L_z}}, \quad (21)$$

$$\frac{(D_1)_0 h_0(\theta_1)_0}{L_z} D_1 h\frac{\partial\theta_1}{\partial z}\bigg|_{z=\frac{m-0}{L_z}} = \frac{(D_2)_0 H_0(\theta_2)_0}{L_z} D_2 H\frac{\partial\theta_2}{\partial z}\bigg|_{y=\frac{m+0}{L_z}}. \quad (22)$$

In developing an effective numerical algorithm for predicting changes in the salt concentration in groundwater and pressure water, an undisclosed finite differential scheme with second-order accuracy $O[(\Delta x)^2 + (\Delta z)^2 + (\Delta\tau)^2]$ is used [18-23]. To do this, we introduce a grid with steps $\Delta x, \Delta z, \Delta\tau$ in the domain $D=\{0\le x<L_x,\, 0\le z<L_z,\, 1\le t\le N\}$, considering boundary conditions:

$\omega_{\Delta x,\Delta z,\Delta\tau}=\{(x_i,z_j,t_n),\ x_i=i\Delta x;\ i=0,1,2,...I; z_j=j\Delta z;\ j=0,1,2,...J;\ t_n=n\Delta\tau;\ n=1,2,...,N\}$ We approximate the system (16 *) and boundary conditions (17) - (22) using a grid $\omega_{\Delta x,\Delta z,\Delta\tau}$ in $n+\frac{1}{2}$ layer over time along the Ox direction:

$$\left.\begin{aligned}
&\left(\frac{\varphi}{\Delta x^2}+\frac{\varphi_2 h_{i,j}^{n+\frac{1}{2}}}{2\Delta x}(|v_x|+v_x)\right)(\theta_1)_{i-1,j}^{n+\frac{1}{2}}-\left(\frac{h_{i,j}^{n+\frac{1}{2}}}{0.5\Delta\tau}+\frac{\varphi}{\Delta x^2}((D_1)_{i-0.5,j}+(D_1)_{i+0.5,j})+\frac{\varphi_2 h_{i,j}^{n+\frac{1}{2}}}{\Delta x}\right)(\theta_1)_{i,j}^{n+\frac{1}{2}}+\\
&\qquad+\left(\frac{\varphi}{\Delta x^2}(D_1)_{i+0.5,j}-\frac{\varphi_2 h_{i,j}^{n+\frac{1}{2}}}{2\Delta x}(|v_x|-v_x)\right)(\theta_1)_{i+1,j}^{n+\frac{1}{2}}=\\
&=-\left(\frac{h_{i,j}^{n+\frac{1}{2}}}{0.5\Delta\tau}(\theta_1)_{i,j}^{n}+\varphi_1\frac{(D_1)_{i,j-0.5}(\theta_1)_{i,j-1}^{n}-((D_1)_{i,j-0.5}+(D_1)_{i,j+0.5})(\theta_1)_{i,j}^{n}+(D_1)_{i,j+0.5}(\theta_1)_{i,j+1}^{n}}{\Delta z^2}-\right.\\
&\qquad\left.-\frac{\varphi_3 h_{i,j}^{n}}{2\Delta z}(2v_z(\theta_1)_{i,j}^{n}-(|v_z|+v_z)(\theta_1)_{i,j-1}^{n}+(|v_z|-v_z)(\theta_1)_{i,j+1}^{n})+\varphi_4 f\theta_f\right),\\
&\left(\frac{\psi}{\Delta x^2}+\frac{\psi_2 H_{i,j}^{n+\frac{1}{2}}}{2\Delta x}(|v_x|+v_x)\right)(\theta_2)_{i-1,j}^{n+\frac{1}{2}}-\left(\frac{H_{i,j}^{n+\frac{1}{2}}}{0.5\Delta\tau}+\frac{\psi}{\Delta x^2}((D_2)_{i-0.5,j}+(D_2)_{i+0.5,j})+\right.\\
&\qquad\left.+\frac{\psi_2 H_{i,j}^{n+\frac{1}{2}}}{\Delta x}v_x\right)(\theta_2)_{i,j}^{n+\frac{1}{2}}+\left(\frac{\psi}{\Delta x^2}(D_2)_{i+0.5,j}-\frac{\psi_2 H_{i,j}^{n+\frac{1}{2}}}{2\Delta x}(|v_x|-v_x)\right)(\theta_2)_{i+1,j}^{n+\frac{1}{2}}=\\
&-\left(\frac{H_{i,j}^{n+\frac{1}{2}}}{0.5\Delta\tau}(\theta_2)_{i,j}^{n}+\psi_1\frac{(D_2)_{i,j-0.5}(\theta_2)_{i,j-1}^{n}-((D_2)_{i,j-0.5}+(D_2)_{i,j+0.5})(\theta_2)_{i,j}^{n}+(D_2)_{i,j+0.5}(\theta_2)_{i,j+1}^{n}}{\Delta z^2}-\right.\\
&\qquad\left.-\frac{\psi_3 H_{i,j}^{n}}{2\Delta z}(2v_y(\theta_2)_{i,j}^{n}-(|v_z|+v_z)(\theta_2)_{i,j-1}^{n}+(|v_z|-v_z)(\theta_2)_{i,j+1}^{n})+\psi_4 f_1\theta_{1f}\right).
\end{aligned}\right\}\quad(23)$$

Having conducted some transformations and grouping similar terms, the finite-difference system (23) is rewritten in the following form:

$$\tilde{a}_{i,j}(\theta_1)_{i-1,j}^{n+\frac{1}{2}}-\tilde{b}_{i,j}(\theta_1)_{i,j}^{n+\frac{1}{2}}+\tilde{c}_{i,j}(\theta_1)_{i+1,j}^{n+\frac{1}{2}}=-\tilde{d}_{i,j}^{n},\quad(24)$$

$$\tilde{\tilde{a}}_{i,j}(\theta_2)_{i-1,j}^{n+\frac{1}{2}}-\tilde{\tilde{b}}_{i,j}(\theta_2)_{i,j}^{n+\frac{1}{2}}+\tilde{\tilde{c}}_{i,j}(\theta_2)_{i+1,j}^{n+\frac{1}{2}}=-\tilde{\tilde{d}}_{i,j}^{n},\quad(25)$$

here

$$\tilde{a}_{i,j}=\frac{\varphi}{\Delta x^2}+\frac{\varphi_2 h_{i,j}^{n+\frac{1}{2}}}{2\Delta x}(|v_x|+v_x),\quad \tilde{b}_{i,j}=\frac{h_{i,j}^{n+\frac{1}{2}}}{0.5\Delta\tau}+\frac{\varphi}{\Delta x^2}((D_1)_{i-0.5,j}+(D_1)_{i+0.5,j})+\frac{\varphi_2 h_{i,j}^{n+\frac{1}{2}}}{\Delta x}v_x,$$

$$\tilde{c}_{i,j}=\frac{\varphi}{\Delta x^2}(D_1)_{i+0.5,j}-\frac{\varphi_2 h_{i,j}^{n+\frac{1}{2}}}{2\Delta x}(|v_x|-v_x),\quad D_{i-0.5,j}=D_{i-0.5,j}h_{i-0.5,j}^{n+\frac{1}{2}},\ D_{i+0.5,j}=D_{i+0.5,j}h_{i+0.5,j}^{n+\frac{1}{2}},$$

$$\tilde{d}_{i,j}^n=\frac{h_{i,j}^{n+\frac{1}{2}}}{0.5\Delta\tau}(\theta_1)_{i,j}^n+\varphi_1\frac{(D_1)_{i,j-0.5}(\theta_1)_{i,j-1}^n-((D_1)_{i,j-0.5}+(D_1)_{i,j+0.5})(\theta_1)_{i,j}^n+(D_1)_{i,j+0.5}(\theta_1)_{i,j+1}^n}{\Delta z^2}-$$

$$-\frac{\varphi_3 h_{i,j}^n}{2\Delta z}(2v_y(\theta_1)_{i,j}^n-(|v_z|+v_z)(\theta_1)_{i,j-1}^n+(|v_z|-v_z)(\theta_1)_{i,j+1}^n)+\varphi_4 f\cdot\theta_f,$$

$$\tilde{\tilde{a}}_{i,j}=\frac{\psi}{\Delta x^2}+\frac{\psi_2 H_{i,j}^{n+\frac{1}{2}}}{2\Delta x}(|v_x|+v_x),\quad \tilde{\tilde{b}}_{i,j}=\frac{H_{i,j}^{n+\frac{1}{2}}}{0.5\Delta\tau}+\frac{\psi}{\Delta x^2}((D_2)_{i-0.5,j}+(D_2)_{i+0.5,j})+\frac{\psi_2 H_{i,j}^{n+\frac{1}{2}}}{\Delta x}v_x,$$

$$\tilde{\tilde{c}}_{i,j}=\frac{\psi}{\Delta x^2}(D_2)_{i+0.5,j}-\frac{\psi_2 H_{i,j}^{n+\frac{1}{2}}}{2\Delta x}(|v_x|-v_x),\quad D_{i-0.5,j}=D_{i-0.5,j}H_{i-0.5,j}^{n+\frac{1}{2}},\ D_{i+0.5,j}=D_{i+0.5,j}H_{i+0.5,j}^{n+\frac{1}{2}},$$

$$\tilde{\tilde{d}}_{i,j}^n=\frac{H_{i,j}^{n+\frac{1}{2}}}{0.5\Delta\tau}(\theta_2)_{i,j}^n+\psi_1\frac{(D_2)_{i,j-0.5}(\theta_2)_{i,j-1}^n-((D_2)_{i,j-0.5}+(D_2)_{i,j+0.5})(\theta_2)_{i,j}^n+(D_2)_{i,j+0.5}(\theta_2)_{i,j+1}^n}{\Delta z^2}-$$

$$-\frac{\psi_3 H_{i,j}^n}{2\Delta z}(2v_y(\theta_2)_{i,j}^n-(|v_z|+v_z)(\theta_2)_{i,j-1}^n+(|v_z|-v_z)(\theta_2)_{i,j+1}^n)+\psi_4 f_1\cdot\theta_{1f}.$$

The resulting systems of equations (24) and (25) concerning the sought-for variables are solved by the sweep method, where the sweep coefficients are calculated as:

$$(\theta_1)_{i,j}^{n+\frac{1}{2}}=\tilde{\alpha}_{i+1,j}(\theta_1)_{i+1,j}^{n+\frac{1}{2}}+\tilde{\beta}_{i+1,j}^n,\tag{26}$$

$$(\theta_2)_{i,j}^{n+\frac{1}{2}}=\tilde{\tilde{\alpha}}_{i+1,j}(\theta_2)_{i+1,j}^{n+\frac{1}{2}}+\tilde{\tilde{\beta}}_{i+1,j}^n.\tag{27}$$

Here $\tilde{\alpha}_{i,j}$, $\tilde{\beta}_{i,j}^n$, $\tilde{\tilde{\alpha}}_{i,j}$, $\tilde{\tilde{\beta}}_{i,j}^n$ are the sweep coefficients

$$\tilde{\alpha}_{i+1,j}=\frac{\tilde{c}_{i,j}}{\tilde{b}_{i,j}-\tilde{a}_{i,j}\tilde{\alpha}_{i,j}},\quad \tilde{\beta}_{i+1,j}^n=\frac{\tilde{d}_{i,j}^n+\tilde{a}_{i,j}\tilde{\beta}_{i,j}^n}{\tilde{b}_{i,j}-\tilde{a}_{i,j}\tilde{\alpha}_{i,j}},\quad \tilde{\tilde{\alpha}}_{i,j}=\frac{\tilde{\tilde{c}}_{i,j}}{\tilde{\tilde{b}}_{i,j}-\tilde{\tilde{a}}_{i,j}\tilde{\tilde{\alpha}}_{i,j}},\ ,\tilde{\tilde{\beta}}_{i+1,j}^n=\frac{\tilde{\tilde{d}}_{i,j}^n+\tilde{\tilde{a}}_{i,j}\tilde{\tilde{\beta}}_{i,j}^n}{\tilde{\tilde{b}}_{i,j}-\tilde{\tilde{a}}_{i,j}\tilde{\tilde{\alpha}}_{i,j}}.$$

Now, accordingly, we approximate the boundary conditions (17)-(22) and obtain:

$$\frac{\mu_1 h_0(\theta_1)_0}{L_x}h_{1,j}^{n+\frac{1}{2}}\frac{(\theta_1)_{0,j}^{n+\frac{1}{2}}-4(\theta_1)_{1,j}^{n+\frac{1}{2}}+3(\theta_1)_{2,j}^{n+\frac{1}{2}}}{2\Delta x}=-((\theta_1)_0(\theta_1)_{1,j}^{n+\frac{1}{2}}-(\theta_1)_0),\tag{28}$$

$$\frac{\mu_1 h_0(\theta_1)_0}{L_x}h_{I,j}^{n+\frac{1}{2}}\frac{-3(\theta_1)_{I-1,j}^{n+\frac{1}{2}}+4(\theta_1)_{I,j}^{n+\frac{1}{2}}-(\theta_1)_{I+1,j}^{n+\frac{1}{2}}}{2\Delta x}=((\theta_1)_0(\theta_1)_{I,j}^{n+\frac{1}{2}}-(\theta_1)_0),\tag{29}$$

$$\frac{\mu_1 h_0(\theta_1)_0}{L_z}h_{i,1}^{n+1}\frac{(\theta_1)_{i,0}^{n+1}-4(\theta_1)_{i,1}^{n+1}+3(\theta_1)_{i,2}^{n+1}}{2\Delta z}=-((\theta_1)_0(\theta_1)_{i,1}^{n+1}-(\theta_1)_0),\tag{30}$$

$$\frac{\mu_1 h_0(\theta_1)_0}{L_z}h_{i,J}^{n+1}\frac{-3(\theta_1)_{i,J-1}^{n+1}+4(\theta_1)_{i,J}^{n+1}-(\theta_1)_{i,J+1}^{n+1}}{2\Delta z}=((\theta_1)_0(\theta_1)_{i,J}^{n+1}-(\theta_1)_0),\tag{31}$$

$$\frac{\mu_2 H_0(\theta_2)_0}{L_x}H_{1,j}^{n+\frac{1}{2}}\frac{(\theta_2)_{0,j}^{n+\frac{1}{2}}-4(\theta_2)_{1,j}^{n+\frac{1}{2}}+3(\theta_2)_{2,j}^{n+\frac{1}{2}}}{2\Delta x}=-((\theta_2)_0(\theta_2)_{1,j}^{n+\frac{1}{2}}-(\theta_2)_0),\tag{32}$$

$$\frac{\mu_2 H_0(\theta_2)_0}{L_x} H_{I,j}^{n+\frac{1}{2}} \frac{-3(\theta_2)_{I-1,j}^{n+\frac{1}{2}} + 4(\theta_2)_{I,j}^{n+\frac{1}{2}} - (\theta_2)_{I+1,j}^{n+\frac{1}{2}}}{2\Delta x} = ((\theta_2)_0(\theta_2)_{I,j}^{n+\frac{1}{2}} - (\theta_2)_0), \tag{33}$$

$$\frac{\mu_2 H_0(\theta_2)_0}{L_z} H_{i,1}^{n+1} \frac{(\theta_2)_{i,0}^{n+1} - 4(\theta_2)_{i,1}^{n+1} + 3(\theta_2)_{i,2}^{n+1}}{2\Delta z} = -((\theta_2)_0(\theta_2)_{i,1}^{n+1} - (\theta_2)_0), \tag{34}$$

$$\frac{\mu_2 H_0(\theta_2)_0}{L_z} H_{i,J}^{n+1} \frac{-3(\theta_2)_{i,J-1}^{n+1} + 4(\theta_2)_{i,J}^{n+1} - (\theta_2)_{i,J+1}^{n+1}}{2\Delta z} = ((\theta_2)_0(\theta_2)_{i,J}^{n+1} - (\theta_2)_0), \tag{35}$$

$$(\theta_1)_0(\theta_1)_{i,\frac{J}{2}}^n = (\theta_2)_0(\theta_2)_{i,\frac{J}{2}}^n, \tag{36}$$

$$\begin{gathered} \frac{(D_1)_0 h_0(\theta_1)_0}{L_z}(D_1)_{i,J} h_{i,J}^n \frac{-3(\theta_1)_{i,J-1}^n + 4(\theta_1)_{i,J}^n - (\theta_1)_{i,J+1}^n}{2\Delta z} = \\ = \frac{(D_2)_0 H_0(\theta_2)_0}{L_z}(D_2)_{i,J} H_{i,J}^n \frac{-3(\theta_2)_{i,J-1}^n + 4(\theta_2)_{i,J}^n - (\theta_2)_{i,J+1}^n}{2\Delta z}. \end{gathered} \tag{37}$$

If $i=1$, then instead of equations (24) and (25), we obtain:

$$(\theta_1)_{2,j}^{n+\frac{1}{2}} = -\frac{\tilde{a}_{i,j}}{\tilde{c}_{i,j}}(\theta_1)_{0,j}^{n+\frac{1}{2}} + \frac{\tilde{b}_{i,j}}{\tilde{c}_{i,j}}(\theta_1)_{1,j}^{n+\frac{1}{2}} - \frac{\tilde{d}_{i,j}^n}{\tilde{c}_{i,j}}, \tag{38}$$

$$(\theta_2)_{2,j}^{n+\frac{1}{2}} = -\frac{\tilde{\tilde{a}}_{i,j}}{\tilde{\tilde{c}}_{i,j}}(\theta_2)_{0,j}^{n+\frac{1}{2}} + \frac{\tilde{\tilde{b}}_{i,j}}{\tilde{\tilde{c}}_{i,j}}(\theta_2)_{1,j}^{n+\frac{1}{2}} - \frac{\tilde{\tilde{d}}_{i,j}^n}{\tilde{\tilde{c}}_{i,j}}, \tag{39}$$

With (28) and (32), by simplifying the boundary conditions, we obtain:

$$(\theta_1)_{2,j}^{n+\frac{1}{2}} = \frac{4\mu_1 h_0 h_{1,j}^{n+\frac{1}{2}} - 2\Delta x L_x}{\mu_1 h_0 h_{1,j}^{n+\frac{1}{2}}}(\theta_1)_{1,j}^{n+\frac{1}{2}} - \frac{1}{3}(\theta_1)_{0,j}^{n+\frac{1}{2}} + \frac{2\Delta x L_x}{\mu_1 h_0 h_{1,j}^{n+\frac{1}{2}}}, \tag{40}$$

$$(\theta_2)_{2,j}^{n+\frac{1}{2}} = -3(\theta_2)_{0,j}^{n+\frac{1}{2}} + \frac{4\mu_2 H_0 H_{1,j}^{n+\frac{1}{2}} - 2\Delta x L_x}{\mu_2 H_0 H_{1,j}^{n+\frac{1}{2}}}(\theta_2)_{1,j}^{n+\frac{1}{2}} + \frac{2\Delta x L_x}{\mu_2 H_0 H_{1,j}^{n+\frac{1}{2}}}. \tag{41}$$

Comparing (38) with (40) and (39) with (41), we find $(\theta_1)_{0,j}^{n+\frac{1}{2}}$ and $(\theta_2)_{0,j}^{n+\frac{1}{2}}$:

$$(\theta_1)_{0,j}^{n+\frac{1}{2}} = \frac{\mu_1 h_0 h_{1,j}^{n+\frac{1}{2}}\tilde{b}_{i,j} - 4\mu_1 h_0 h_{1,j}^{n+\frac{1}{2}}\tilde{c}_{i,j} + 2\Delta x L_x \tilde{c}_{i,j}}{\mu_1 h_0 h_{1,j}^{n+\frac{1}{2}}(3\tilde{a}_{i,j} - \tilde{c}_{i,j})}(\theta_1)_{1,j}^{n+\frac{1}{2}} - \frac{\mu_1 h_0 h_{1,j}^{n+\frac{1}{2}}\tilde{d}_{i,j}^n - 2\Delta x L_x \tilde{c}_{i,j}}{\mu_1 h_0 h_{1,j}^{n+\frac{1}{2}}(3\tilde{a}_{i,j} - \tilde{c}_{i,j})}, \tag{42}$$

$$(\theta_2)_{0,j}^{n+\frac{1}{2}} = \frac{3\mu_2 H_0 H_{1,j}^{n+\frac{1}{2}}\tilde{\tilde{b}}_{i,j} - 4\mu_2 H_0 H_{1,j}^{n+\frac{1}{2}}\tilde{\tilde{c}}_{i,j} + 2\Delta x L_x \tilde{\tilde{c}}_{i,j}}{\mu_2 H_0 H_{1,j}^{n+\frac{1}{2}}(3\tilde{\tilde{a}}_{i,j} - \tilde{\tilde{c}}_{i,j})}(\theta_2)_{1,j}^{n+\frac{1}{2}} - \frac{3\mu_2 H_0 H_{1,j}^{n+\frac{1}{2}}\tilde{\tilde{d}}_{i,j}^n + 2\Delta x L_x \tilde{\tilde{c}}_{i,j}}{\mu_2 H_0 H_{1,j}^{n+\frac{1}{2}}(3\tilde{\tilde{a}}_{i,j} - \tilde{\tilde{c}}_{i,j})}. \tag{43}$$

(26) and (27) are recurrent formulas, and if we assume that $i=0$, then we have the following equations:

$$(\theta_1)_{0,j}^{n+\frac{1}{2}} = \tilde{\alpha}_{1,j}(\theta_1)_{1,j}^{n+\frac{1}{2}} + \tilde{\beta}_{1,j}^n, \tag{44}$$

$$(\theta_2)_{0,j}^{n+\frac{1}{2}} = \tilde{\tilde{\alpha}}_{1,j}(\theta_2)_{1,j}^{n+\frac{1}{2}} + \tilde{\tilde{\beta}}_{1,j}^{n}. \tag{45}$$

Comparing equations (42) and (44) and (43) and (45), we find the initial values of coefficients $\tilde{\alpha}_{1,j}$, $\tilde{\beta}_{1,j}^n$ and $\tilde{\tilde{\alpha}}_{1,j}$, $\tilde{\tilde{\beta}}_{1,j}^n$:

$$\tilde{\alpha}_{1,j} = \frac{\mu_1 h_0 h_{1,j}^{n+\frac{1}{2}}\tilde{b}_{i,j} - 4\mu_1 h_0 h_{1,j}^{n+\frac{1}{2}}\tilde{c}_{i,j} + 2\Delta x L_x \tilde{c}_{i,j}}{\mu_1 h_0 h_{1,j}^{n+\frac{1}{2}}(3\tilde{a}_{i,j} - \tilde{c}_{i,j})}, \quad \tilde{\beta}_{1,j}^n = -\frac{\mu_1 h_0 h_{1,j}^{n+\frac{1}{2}}\tilde{d}_{i,j}^n - 2\Delta x L_x \tilde{c}_{i,j}}{\mu_1 h_0 h_{1,j}^{n+\frac{1}{2}}(3\tilde{a}_{i,j} - \tilde{c}_{i,j})},$$

$$\tilde{\tilde{\alpha}}_{1,j} = \frac{3\mu_2 H_0 H_{1,j}^{n+\frac{1}{2}}\tilde{\tilde{b}}_{i,j} - 4\mu_2 H_0 H_{1,j}^{n+\frac{1}{2}}\tilde{\tilde{c}}_{i,j} + 2\Delta x L_x \tilde{\tilde{c}}_{i,j}}{\mu_2 H_0 H_{1,j}^{n+\frac{1}{2}}(3\tilde{\tilde{a}}_{i,j} - \tilde{\tilde{c}}_{i,j})}, \quad \tilde{\tilde{\beta}}_{1,j}^n = -\frac{3\mu_2 H_0 H_{1,j}^{n+\frac{1}{2}}\tilde{\tilde{d}}_{i,j}^n + 2\Delta x L_x \tilde{\tilde{c}}_{i,j}}{\mu_2 H_0 H_{1,j}^{n+\frac{1}{2}}(3\tilde{\tilde{a}}_{i,j} - \tilde{\tilde{c}}_{i,j})}.$$

If we assume that $i = I$, then from systems (24) and (25)

$$(\theta_1)_{I+1,j}^{n+\frac{1}{2}} = -\frac{\tilde{a}_{I,j}}{\tilde{c}_{I,j}}(\theta_1)_{I-1,j}^{n+\frac{1}{2}} + \frac{\tilde{b}_{I,j}}{\tilde{c}_{I,j}}(\theta_1)_{I,j}^{n+\frac{1}{2}} - \frac{\tilde{d}_{I,j}^n}{\tilde{c}_{I,j}}, \tag{46}$$

$$(\theta_2)_{I+1,j}^{n+\frac{1}{2}} = -\frac{\tilde{\tilde{a}}_{I,j}}{\tilde{\tilde{c}}_{I,j}}(\theta_2)_{I-1,j}^{n+\frac{1}{2}} + \frac{\tilde{\tilde{b}}_{I,j}}{\tilde{\tilde{c}}_{I,j}}(\theta_2)_{I,j}^{n+\frac{1}{2}} - \frac{\tilde{\tilde{d}}_{I,j}^n}{\tilde{\tilde{c}}_{I,j}}, \tag{47}$$

with equations (29) and (33) by simplifying the boundary conditions, we have the following equations:

$$(\theta_1)_{I+1,j}^{n+\frac{1}{2}} = -3(\theta_1)_{I-1,j}^{n+\frac{1}{2}} + \frac{4\mu_1 h_0 h_{I,j}^{n+\frac{1}{2}} - 2\Delta x L_x}{\mu_1 h_0 h_{I,j}^{n+\frac{1}{2}}}(\theta_1)_{I,j}^{n+\frac{1}{2}} + \frac{2\Delta x L_x}{\mu_1 h_0 h_{I,j}^{n+\frac{1}{2}}}, \tag{48}$$

$$(\theta_2)_{I+1,j}^{n+\frac{1}{2}} = 3(\theta_2)_{I-1,j}^{n+\frac{1}{2}} - \frac{4\mu_2 H_0 H_{I,j}^{n+\frac{1}{2}} - 2\Delta x L_x}{\mu_2 H_0 H_{I,j}^{n+\frac{1}{2}}}(\theta_2)_{I,j}^{n+\frac{1}{2}} + \frac{2\Delta x L_x}{\mu_2 H_0 H_{I,j}^{n+\frac{1}{2}}}, \tag{49}$$

Comparing equations (46) and (48) and (47) and (49), we find $(\theta_1)_{I-1,j}^{n+\frac{1}{2}}$ and $(\theta_2)_{I-1,j}^{n+\frac{1}{2}}$:

$$(\theta_1)_{I-1,j}^{n+\frac{1}{2}} = \frac{\mu_1 h_0 h_{I,j}^{n+\frac{1}{2}}\tilde{b}_{I,j} - 4\mu_1 h_0 h_{I,j}^{n+\frac{1}{2}}\tilde{c}_{I,j} + 2\Delta x L_x \tilde{c}_{I,j}}{3\tilde{c}_{I,j} - \tilde{a}_{I,j}}(\theta_1)_{I,j}^{n+\frac{1}{2}} - \frac{\mu_1 h_0 h_{I,j}^{n+\frac{1}{2}}\tilde{d}_{I,j}^n + 2\Delta x L_x \tilde{c}_{I,j}}{3\tilde{c}_{I,j} - \tilde{a}_{I,j}}, \tag{50*}$$

$$(\theta_2)_{I-1,j}^{n+\frac{1}{2}} = \frac{\mu_2 H_0 H_{I,j}^{n+\frac{1}{2}}\tilde{\tilde{b}}_{I,j} + 4\mu_2 H_0 H_{I,j}^{n+\frac{1}{2}}\tilde{\tilde{c}}_{I,j} - 2\Delta x L_x \tilde{\tilde{c}}_{I,j}}{\mu_2 H_0 H_{I,j}^{n+\frac{1}{2}}(\tilde{\tilde{a}}_{I,j} + 3\tilde{\tilde{c}}_{I,j})}(\theta_2)_{I,j}^{n+\frac{1}{2}} - \frac{2\Delta x L_x \tilde{\tilde{c}}_{I,j} + \mu_2 H_0 H_{I,j}^{n+\frac{1}{2}}\tilde{\tilde{d}}_{I,j}^n}{\mu_2 H_0 H_{I,j}^{n+\frac{1}{2}}(\tilde{\tilde{a}}_{I,j} + 3\tilde{\tilde{c}}_{I,j})}. \tag{50}$$

(26) and (27) are in recurrent formulas, and if we assume that $i = I - 1$, then we have the following equations:

$$(\theta_1)_{I-1,j}^{n+\frac{1}{2}} = \tilde{\alpha}_{I,j}(\theta_1)_{I,j}^{n+\frac{1}{2}} + \tilde{\beta}_{I,j}^n, \tag{51}$$

$$(\theta_2)_{I-1,j}^{n+\frac{1}{2}} = \tilde{\tilde{\alpha}}_{I,j}(\theta_2)_{I,j}^{n+\frac{1}{2}} + \tilde{\tilde{\beta}}_{I,j}^{n}. \qquad (52)$$

Comparing equations (50 *) and (51) and (50) and (52), we find the $(\theta_1)_{I,j}^{n+\frac{1}{2}}$ and $(\theta_2)_{I,j}^{n+\frac{1}{2}}$ boundary values of the salt concentrations in the aquifer and pressure aquifer:

$$(\theta_1)_{I,j}^{n+\frac{1}{2}} = \frac{\mu_1 h_0 h_{I,j}^{n+\frac{1}{2}}\tilde{d}_{I,j}^{n} + 2\Delta x L_x \tilde{c}_{I,j} + (3\tilde{c}_{I,j} - \tilde{a}_{I,j})\tilde{\beta}_{I,j}^{n}}{\mu_1 h_0 h_{I,j}^{n+\frac{1}{2}}\tilde{b}_{I,j} - 4\mu_1 h_0 h_{I,j}^{n+\frac{1}{2}}\tilde{c}_{I,j} + 2\Delta x L_x \tilde{c}_{I,j} - (3\tilde{c}_{I,j} - \tilde{a}_{I,j})\tilde{\alpha}_{I,j}},$$

$$(\theta_2)_{I,j}^{n+\frac{1}{2}} = \frac{\mu_2 H_0 H_{I,j}^{n+\frac{1}{2}}(\tilde{\tilde{a}}_{I,j} + 3\tilde{\tilde{c}}_{I,j})\tilde{\tilde{\beta}}_{I,j}^{n} + 2\Delta x L_x \tilde{\tilde{c}}_{I,j} + \mu_2 H_0 H_{I,j}^{n+\frac{1}{2}}\tilde{\tilde{d}}_{I,j}^{n}}{\mu_2 H_0 H_{I,j}^{n+\frac{1}{2}}\tilde{\tilde{b}}_{I,j} + 4\mu_2 H_0 H_{I,j}^{n+\frac{1}{2}}\tilde{\tilde{c}}_{I,j} - 2\Delta x L_x \tilde{\tilde{c}}_{I,j} - \mu_2 H_0 H_{I,j}^{n+\frac{1}{2}}(\tilde{\tilde{a}}_{I,j} + 3\tilde{\tilde{c}}_{I,j})\tilde{\tilde{\alpha}}_{I,j}}.$$

We approximate the system (16 *) using an implicit scheme in the grid $\omega_{\Delta x,\Delta z,\Delta\tau}$ in $n+1$ layer over time along the Oz direction, like the above algorithm, and bring it to the following system of three diagonal algebraic equations:

$$\overline{\overline{\overline{a}}}_{i,j}(\theta_1)_{i,j-1}^{n+1} - \overline{\overline{\overline{b}}}_{i,j}(\theta_1)_{i,j}^{n+1} + \overline{\overline{\overline{c}}}_{i,j}(\theta_1)_{i,j+1}^{n+1} = -\overline{\overline{\overline{d}}}_{i,j}^{n}, \qquad (53)$$

$$\tilde{\tilde{a}}_{i,j}(\theta_2)_{i,j-1}^{n+1} - \tilde{\tilde{b}}_{i,j}(\theta_2)_{i,j}^{n+1} + \tilde{\tilde{c}}_{i,j}(\theta_2)_{i,j+1}^{n+1} = -\tilde{\tilde{d}}_{i,j}^{n}, \qquad (54)$$

here

$$\overline{\overline{\overline{a}}}_{i,j} = \frac{\varphi_1}{\Delta z^2}(D_1)_{i,j-0.5} + \frac{\varphi_3 h_{i,j}^{n+1}}{2\Delta z}(|v_z| + v_z), \quad \overline{\overline{\overline{c}}}_{i,j} = \frac{\varphi_1}{\Delta z^2}(D_1)_{i,j+0.5} - \frac{\varphi_3 h_{i,j}^{n+1}}{2\Delta z}(|v_z| - v_z),$$

$$\overline{\overline{\overline{b}}}_{i,j} = \frac{h_{i,j}^{n+1}}{0.5\Delta\tau} + \frac{\varphi_1}{\Delta z^2}((D_1)_{i,j-0.5} + (D_1)_{i,j+0.5}) + \frac{\varphi_3 h_{i,j}^{n+1}}{\Delta z}v_z,$$

$$D_{i,j-0.5} = D_{i,j-0.5}h_{i,j-0.5}^{n+1}, \quad D_{i,j+0.5} = D_{i,j+0.5}h_{i,j+0.5}^{n+1},$$

$$\overline{\overline{\overline{d}}}_{i,j}^{n} = \frac{h_{i,j}^{n+1}}{0.5\Delta\tau}(\theta_1)_{i,j}^{n+\frac{1}{2}} + \varphi\frac{(D_1)_{i-0.5,j}(\theta_1)_{i-1,j}^{n+\frac{1}{2}} - ((D_1)_{i-0.5,j} + (D_1)_{i+0.5,j})(\theta_1)_{i,j}^{n+\frac{1}{2}} + (D_1)_{i+0.5,j}(\theta_1)_{i+1,j}^{n+\frac{1}{2}}}{\Delta x^2} +$$

$$+\frac{\varphi_2 h_{i,j}^{n+\frac{1}{2}}}{2\Delta x}\left(2v_x(\theta_1)_{i,j}^{n+\frac{1}{2}} - (|v_x| + v_x)(\theta_1)_{i-1,j}^{n+\frac{1}{2}} + (|v_x| - v_x)(\theta_1)_{i+1,j}^{n+\frac{1}{2}}\right) + \varphi_4 f \cdot \theta_f,$$

$$\tilde{\tilde{a}}_{i,j} = \frac{\psi_1}{\Delta z^2}(D_2)_{i,j-0.5} + \frac{\psi_3 H_{i,j}^{n+1}}{2\Delta z}(|v_z| + v_z), \quad \tilde{\tilde{c}}_{i,j} = \frac{\psi_1}{\Delta z^2}(D_2)_{i,j+0.5} - \frac{\psi_3 H_{i,j}^{n+1}}{2\Delta z}(|v_z| - v_z),$$

$$\tilde{\tilde{b}}_{i,j} = \frac{H_{i,j}^{n+1}}{0.5\Delta\tau} + \frac{\psi_1}{\Delta z^2}((D_2)_{i,j-0.5} + (D_2)_{i,j+0.5}) + \frac{\psi_3 H_{i,j}^{n+1}}{\Delta z}v_z,$$

$$D_{i,j-0.5} = D_{i,j-0.5}H_{i,j-0.5}^{n+1}, \quad D_{i,j+0.5} = D_{i,j+0.5}H_{i,j+0.5}^{n+1},$$

$$\tilde{\tilde{d}}_{i,j}^{n} = \frac{H_{i,j}^{n+1}}{0.5\Delta\tau}(\theta_2)_{i,j}^{n+\frac{1}{2}} + \psi\frac{(D_2)_{i-0.5,j}(\theta_2)_{i-1,j}^{n+\frac{1}{2}} - ((D_2)_{i-0.5,j} + (D_2)_{i+0.5,j})(\theta_2)_{i,j}^{n+\frac{1}{2}} + (D_2)_{i+0.5,j}(\theta_2)_{i+1,j}^{n+\frac{1}{2}}}{\Delta x^2} +$$

$$+\frac{\psi_2 H_{i,j}^{n+\frac{1}{2}}}{2\Delta x}\left(2v_x(\theta_2)_{i,j}^{n+\frac{1}{2}} - (|v_x| + v_x)(\theta_2)_{i-1,j}^{n+\frac{1}{2}} + (|v_x| - v_x)(\theta_2)_{i+1,j}^{n+\frac{1}{2}}\right) + \psi_4 f_1 \cdot \theta_{1f}.$$

We solve (53) and (54) systems of three diagonal linear algebraic equations sweep method. We look for the solution in the inner points of the network by the following recurrent formulas:

$$(\theta_1)^{n+1}_{i,j} = \bar{\bar{\bar{\alpha}}}_{i,j+1}(\theta_1)^{n+1}_{i,j+1} + \bar{\bar{\bar{\beta}}}^{n}_{i,j+1}, \tag{55}$$

$$(\theta_2)^{n+1}_{i,j} = \tilde{\tilde{\tilde{\alpha}}}_{i,j+1}(\theta_2)^{n+1}_{i,j+1} + \tilde{\tilde{\tilde{\beta}}}^{n}_{i,j+1}, \tag{56}$$

Here $\bar{\bar{\bar{\alpha}}}_{i,j}$, $\bar{\bar{\bar{\beta}}}^{n}_{i,j}$, $\tilde{\tilde{\tilde{\alpha}}}_{i,j}$, $\tilde{\tilde{\tilde{\beta}}}^{n}_{i,j}$ are sweep coefficients, and they are determined by the following formulas:

$$\bar{\bar{\bar{\alpha}}}_{i,j+1} = \frac{\bar{\bar{\bar{c}}}_{i,j}}{\bar{\bar{\bar{b}}}_{i,j} - \bar{\bar{\bar{a}}}_{i,j}\bar{\bar{\bar{\alpha}}}_{i,j}}, \quad \bar{\bar{\bar{\beta}}}^{n}_{i,j+1} = \frac{\bar{\bar{\bar{d}}}^{n}_{i,j} + \bar{\bar{\bar{a}}}_{i,j}\bar{\bar{\bar{\beta}}}^{n}_{i,j}}{\bar{\bar{\bar{b}}}_{i,j} - \bar{\bar{\bar{a}}}_{i,j}\bar{\bar{\bar{\alpha}}}_{i,j}}, \quad \tilde{\tilde{\tilde{\alpha}}}_{i,j+1} = \frac{\tilde{\tilde{\tilde{c}}}_{i,j}}{\tilde{\tilde{\tilde{b}}}_{i,j} - \tilde{\tilde{\tilde{a}}}_{i,j}\tilde{\tilde{\tilde{\alpha}}}_{i,j}}, \quad \tilde{\tilde{\tilde{\beta}}}^{n}_{i,j+1} = \frac{\tilde{\tilde{\tilde{d}}}^{n}_{i,j} + \tilde{\tilde{\tilde{a}}}_{i,j}\tilde{\tilde{\tilde{\beta}}}^{n}_{i,j}}{\tilde{\tilde{\tilde{b}}}_{i,j} - \tilde{\tilde{\tilde{a}}}_{i,j}\tilde{\tilde{\tilde{\alpha}}}_{i,j}}.$$

$\bar{\bar{\bar{\alpha}}}_{i,1}$, $\bar{\bar{\bar{\beta}}}^{n}_{i,1}$ and $\tilde{\tilde{\tilde{\alpha}}}_{i,1}$, $\tilde{\tilde{\tilde{\beta}}}^{n}_{i,1}$ are the initial values of the coefficients calculated along the Oz direction from the following equations

$$\bar{\bar{\bar{\alpha}}}_{i,1} = \frac{3\mu_1 h_0 h^{n+1}_{i,1}\bar{\bar{\bar{b}}}_{i,1} - 4\mu_1 h_0 h^{n+1}_{i,1}\bar{\bar{\bar{c}}}_{i,1} + 2\Delta z L_z \bar{\bar{\bar{c}}}_{i,1}}{3\mu_1 h_0 h^{n+1}_{i,1}(3\bar{\bar{\bar{a}}}_{i,1} - \bar{\bar{\bar{c}}}_{i,1})}, \quad \bar{\bar{\bar{\beta}}}^{n}_{i,1} = -\frac{2\Delta z L_z \bar{\bar{\bar{c}}}_{i,1} + 3\mu_1 h_0 h^{n+1}_{i,1}\bar{\bar{\bar{d}}}^{n}_{i,1}}{3\mu_1 h_0 h^{n+1}_{i,1}(3\bar{\bar{\bar{a}}}_{i,1} - \bar{\bar{\bar{c}}}_{i,1})},$$

$$\tilde{\tilde{\tilde{\alpha}}}_{i,1} = \frac{3\mu_2 H_0 H^{n+1}_{i,1}\tilde{\tilde{\tilde{b}}}_{i,1} - 4\mu_2 H_0 H^{n+1}_{i,1}\tilde{\tilde{\tilde{c}}}_{i,1} + 2\Delta z L_z \tilde{\tilde{\tilde{c}}}_{i,1}}{\mu_2 H_0 H^{n+1}_{i,1}(3\tilde{\tilde{\tilde{a}}}_{i,1} - \tilde{\tilde{\tilde{c}}}_{i,1})}, \quad \tilde{\tilde{\tilde{\beta}}}^{n}_{i,1} = -\frac{2\Delta z L_z \tilde{\tilde{\tilde{c}}}_{i,1} + 3\mu_2 H_0 H^{n+1}_{i,1}\tilde{\tilde{\tilde{d}}}^{n}_{i,1}}{\mu_2 H_0 H^{n+1}_{i,1}(3\tilde{\tilde{\tilde{a}}}_{i,1} - \tilde{\tilde{\tilde{c}}}_{i,1})}.$$

Comparing the systems of three diagonal linear algebraic equations (53), (54), recurrent formulas (55), (56) and boundary conditions (34), (35) along the Oz direction, respectively, we find the boundary values of salt concentrations in the subsurface and pressure aquifers:

$$(\theta_1)^{n+1}_{i,J} = \frac{2\Delta z L_z \bar{\bar{\bar{c}}}_{i,J} - \mu_1 h_0 h^{n+1}_{i,J}\bar{\bar{\bar{d}}}^{n}_{i,J} + \mu_1 h_0 h^{n+1}_{i,J}(\bar{\bar{\bar{a}}}_{i,J} - 3\bar{\bar{\bar{c}}}_{i,J})\bar{\bar{\bar{\beta}}}^{n}_{i,J}}{\mu_1 h_0 h^{n+1}_{i,J}\bar{\bar{\bar{b}}}_{i,J} - 4\mu_1 h_0 h^{n+1}_{i,J}\bar{\bar{\bar{c}}}_{i,J} + 2\Delta z L_z \bar{\bar{\bar{c}}}_{i,J} - \mu_1 h_0 h^{n+1}_{i,J}(\bar{\bar{\bar{a}}}_{i,J} - 3\bar{\bar{\bar{c}}}_{i,J})\bar{\bar{\bar{\alpha}}}_{i,J}},$$

$$(\theta_2)^{n+1}_{i,J} = \frac{2\Delta z L_z \tilde{\tilde{\tilde{c}}}_{i,J} + \mu_2 H_0 H^{n+1}_{i,J}\tilde{\tilde{\tilde{d}}}^{n}_{i,J} + \mu_2 H_0 H^{n+1}_{i,J}(\tilde{\tilde{\tilde{a}}}_{i,J} - 3\tilde{\tilde{\tilde{c}}}_{i,J})\tilde{\tilde{\tilde{\beta}}}^{n}_{i,J}}{\mu_2 H_0 H^{n+1}_{i,J}\tilde{\tilde{\tilde{b}}}_{i,J} - 4\mu_2 H_0 H^{n+1}_{i,J}\tilde{\tilde{\tilde{c}}}_{i,J} + 2\Delta z L_z \tilde{\tilde{\tilde{c}}}_{i,J} - \mu_2 H_0 H^{n+1}_{i,J}(\tilde{\tilde{\tilde{a}}}_{i,J} - 3\tilde{\tilde{\tilde{c}}}_{i,J})\tilde{\tilde{\tilde{\alpha}}}_{i,J}}.$$

Using the backward sweep method, the values of the salt concentrations $(\theta_1)^{n+\frac{1}{2}}_{i,J-1}, (\theta_1)^{n+\frac{1}{2}}_{i,J-2}, ..., (\theta_1)^{n+\frac{1}{2}}_{i,1}$, $(\theta_2)^{n+\frac{1}{2}}_{i,J-1}, (\theta_2)^{n+\frac{1}{2}}_{i,J-2}, ..., (\theta_2)^{n+\frac{1}{2}}_{i,1}$ in the sedimentary and pressure aquifers in the $n+\dfrac{1}{2}$ layer over time along the Ox direction, and the values of the salt concentrations $(\theta_1)^{n+1}_{i,J-1}, (\theta_1)^{n+1}_{i,J-2}, ..., (\theta_1)^{n+1}_{i,1}$, $(\theta_2)^{n+1}_{i,J-1}, (\theta_2)^{n+1}_{i,J-2}, ..., (\theta_2)^{n+1}_{i,1}$ in the sedimentary and pressure aquifers in the $n+1$ layer over time along the Oz direction, are found.

3 Results and Discussion

The machine algorithm for solving the problem is as follows:

1st step. Input of initial (baseline) data (input of constants).

2nd step. Calculation of boundary values of the sought for variables from boundary conditions of the problem.

3rd step. Calculation of the elements of a tridiagonal transition matrix obtained by approximating differential operators to finite-difference ones.

4th step. Calculation of sweep coefficients.

5th step. Calculation of values of the sought for variables of the task.

6th step. Adequacy verification of the task.

7th step. Interpretation of computer experiments results.

As a result of numerical computational experiments based on computational algorithms developed for the numerical solution of complex mathematical models, the following graphs were plotted:

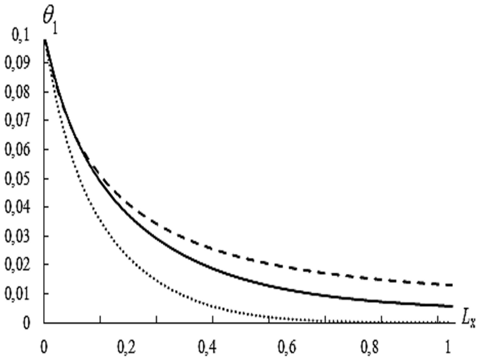

Fig. 1. Variation of salt concentration over time. [Compiled by the authors].

$$\left(\mu_1 = 0.16, \mu_z = 0.01,\ v_x = 0.007, v_z = 0.005,\ D_1 = 0.006\cdot 10^{-2}, D_2 = 0.007\cdot 10^{-2},\ \theta_f = 0.00026, \theta_{1f} = 0.0001\right)$$

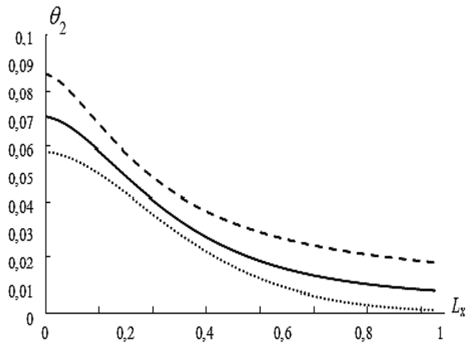

Fig. 2. Variation of salt concentration over time. [Compiled by the authors].

$$\left(\mu_1 = 0.16, \mu_2 = 0.01,\ v_x = 0.007, v_z = 0.005,\ \theta_f = 0.00026, \theta_{1f} = 0.0001,\ D_1 = 0.006\cdot 10^{-2}, D_2 = 0.007\cdot 10^{-2},\right)$$

As a result of changes in the groundwater and pressure water level, we can observe a decrease in the salt concentration in the groundwater and pressure aquifers over time (Figures 1 and 2).

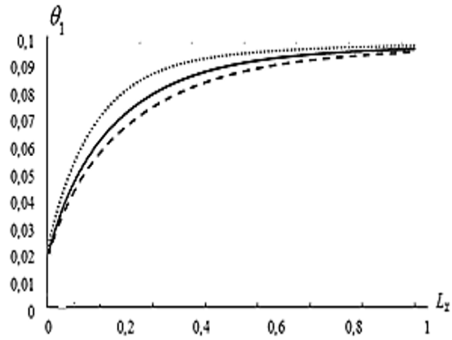

Fig. 3. Variation of salt concentration over time. [Compiled by the authors].
$\left(\mu_1 = 0.6, \mu_2 = 0.1,\ v_x = 0.008, v_z = 0.003,\right.$
$D_1 = 0.004 \cdot 10^{-2}, D_2 = 0.007 \cdot 10^{-2},$
$\left.\theta_f = 0.0036, \theta_{1f} = 0.001\right)$

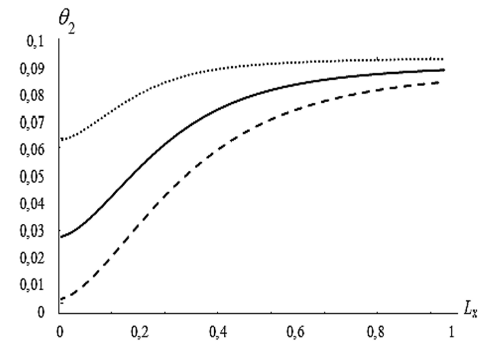

Fig. 4. Variation of salt concentration over time. [Compiled by the authors].
$\left(\mu_1 = 0.6, \mu_2 = 0.1,\ v_x = 0.008, v_z = 0.003,\right.$
$D_1 = 0.004 \cdot 10^{-2}, D_2 = 0.007 \cdot 10^{-2},$
$\left.\theta_f = 0.0036, \theta_{1f} = 0.001\right)$

From the change in the salt concentration of the aquifer, we can see that parameters such as filtration rate components, diffusion coefficients, and salt concentrations (in infiltration water) are important (Figures 1 - 4). The scarcity of water resources in the world is one of the biggest environmental problems. It can be observed that the excessive use of chemicals in industry and agriculture hurts the condition of groundwater, leading to a decrease in the quality of water resources and salinization. In winter and spring, groundwater and pressure water levels rise under the influence of external sources, evaporation, soil porosity in the aquifer, and debits. As a result, we can observe an increase in salt concentrations over time due to changes in parameters such as water loss, filtration rates, and diffusion coefficients, i.e., the occurrence of salinity at the soil surface (Figures 3 and 4). This, in turn, leads to a further reduction in the consumption of water resources and pollution of groundwater. Therefore, studying mathematical and software tools is important in monitoring and predicting the changes in groundwater level and mineralization state. It is shown in this article that the study of important parameters such as functional relationships, porosity, and groundwater is based on nonlinear general boundary conditions. An stable disclosure scheme with a high-order accuracy in time and spatial variables and efficient numerical computational algorithms were developed for solving change problems in the groundwater level. Results from this mathematical model and numerical algorithm are shown in Sections 1 and 4 as graphs. In summer, mineralized waters from flow in the aquifer are discharged into drains. When groundwater enters the well, freshwater is supplied to the water pipeline system. Water is supplied to the sewer network from a low-pressure aquifer. In winter, water flows in the pressure aquifer layer. It should be noted that different groundwater modes lead to the interaction of groundwater's hydrodynamic and hydrochemical modes in both layers. Under such conditions, special attention should be paid to protecting the layer from the ingress of highly mineralized waters from the boundaries of the interacting layers. The research conducted by Wang, XI Chen provided an analysis of mathematical groundwater and surface water flow models based on geofiltration models [11, 13]. F.B. Abutaliev and E.B. Abutaliev developed a mathematical model based on the hydrodynamic nature of the object under study to calculate and predict changes in groundwater levels [2]. E.B. Soboleva proposed a mathematical model for studying salt migration to numerically solve non-stationary problems and monitor the quality of groundwater and surface water [12]. In contrast to the research presented by a system of differential equations (1), an account for the interaction of external factors and evaporations and the changes in groundwater and pressure water levels is very important. The effect of this change on the change in salt concentration in groundwater and pressure water is presented by considering diffusion and convective migration processes. Many scientists addressed these issues, A. Vlasyuk [15], Rong [16], and Wang [13], boundary conditions (2) - (8) were formed that take into account the initial input and output processes.

4 Conclusions

A mathematical model and an efficient numerical algorithm were developed to comprehensively study the geofiltration processes. Computational experiments were conducted to determine the changes in salt concentrations in groundwater and pressure waters. Since the process is characterized by a system of differential equations and corresponding initial and boundary conditions, the method of dimensionless quantities was used. The importance of developing recommendations before introducing new technologies in predicting changes in salt concentration in water depending on the level and flow of wastewater was studied using an effective numerical algorithm constructed. The graphical results presented above show that salinization occurs at the soil surface and that the salinity level is maintained in the remaining intervals. We see that parameters such as filtration rate components, diffusion coefficients, and salt concentrations (in infiltration water) are important for changing the salt concentration in the aquifer. As a result of studying the laws of motion of wastewater flows, it was noted that it is possible to conduct experiments using an algorithm to determine changes in the distance and velocity of wastewater propagation with soluble chemicals and active properties in water. The results show that the model and algorithm can be used to predict the process of water infiltration and changes in the water level in water supply areas, technological processes in salt migration, and qualitative and quantitative analysis of processes in hydrogeology. The created mathematical and numerical apparatus can significantly reduce the volume of full-scale research and minimize experimental work that requires expensive resources in conducting computational experiments on a computer.

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