About one differential model of dynamics of groundwater

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Abstract. When modeling the flow of groundwater and streams together, two different approaches are used, using hydraulic and hydrological models as channel flow models. The former is based on mathematical equations of water movement in open channels. In contrast, the latter is based on simplified empirical and semi-empirical relationships between the hydraulic characteristics of watercourses. In both cases, the watercourse is an internal boundary for the groundwater flow - otherwise, it is advisable to model it as a body of water. The groundwater model can be a scale model or an electrical model of the state of the groundwater or an aquifer. Groundwater models are used to represent the natural flow of groundwater in an environment. Some groundwater models include aspects of groundwater quality. Such groundwater models attempt to predict the fate and movement of a chemical in natural, urban, or hypothetical scenarios. Groundwater models can be used to predict the impact of hydrological changes on aquifer behavior and are often referred to as groundwater simulation models. Also, groundwater models are currently being used in various water management plans for urban areas. Because calculations in mathematical groundwater models are based on groundwater flow equations, which are differential equations that can often only be solved by approximate methods using numerical analysis, these models are also referred to as mathematical, numerical, or computational groundwater models.

1 Introduction

The unsaturated or vadose zone is the primary link between groundwater and hydrological inputs. Soil separates hydrological factors such as precipitation or snowmelt into surface runoff, soil moisture, evapotranspiration, and groundwater recharge. The flows through the unsaturated zone, which link surface water with soil moisture and groundwater, can be ascending or descending, depending on the hydraulic head gradient in the soil, and can be modeled using a numerical solution of the Richards partial differential equation or an ordinary differential. Equation Finite water content method validated for modeling interactions between groundwater and the vadose zone.

Boundary conditions can be related to groundwater levels, artesian pressures, and hydraulic heads along model boundaries on the one hand (head conditions) or to groundwater inflows and outflows along model boundaries on the other. (flow conditions).

This may also include aspects of water quality, such as salinity. The initial conditions refer to the initial values of elements that can increase or decrease over time within the model domain, and they cover basically the same phenomenon as the boundary conditions. Initial and boundary conditions may vary from place to place. Boundary conditions may remain constant or change over time.

The study of boundary value problems with the Poincaré – Tricomi condition for degenerate equations of elliptic and elliptic–hyperbolic types of the second kind, where the characteristics of the equation are also lines of degeneracy, is devoted[5–11]. In this paper, we prove the uniqueness of a solution to a boundary value problem of the Poincare – Tricomi problem type for an elliptic – hyperbolic equation of the second kind describing the groundwater differential model.

2 Statement of the problem

Consider the equation

$$sgny|y|^m u_{xx} + u_{yy} = 0 \qquad (-1 < m < 0) \tag{1}$$

in a domain $D = D_1 \cup D_2$, where D_1 is a simply-connected domain in the plane (x, y), bounded by a curve σ at the first quadrant $(x > 0, y > 0)$ with its end points $A(0,0), B(1,0)$ and with the line segment $AB(y = 0)$ on the real axis *Ox*, and D_2 is a domain bounded by the segment AB and characteristics of equation (1):

$$AC: x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = 0, \quad BC: x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = 1$$

Let us introduce the following notations

$$J = \{(x, y):\ 0 < x < 1,\ y = 0\},\ \partial D = \vec{\sigma} \cup \overline{AB}, \quad 2\beta = \frac{m}{m+2},$$

note that we have

$$-\frac{1}{2} < \beta < 0. \tag{2}$$

3 Formulation of the problem

In the domain D, we consider the following Problem Poincar'e - Tricomi for the equation (1).

Problem. To find a function $u(x, y)$, the following properties:

1) $u(x, y) \in C(\bar{D}) \cup C^1(D \cup \sigma \cup J)$ – and u_x, u_y can tend to infinity of the order less than -2β at points $A(0,0)$ and $B(1,0)$;

2) A function $u(x,y)\in C^2\left(\bar{D}_1\right)$ is a regular solution of equation (1) in the domain $D_1\left(y>0\right)$, and is the generalized solution from the class R_2 [7,8] in the domain $D_2\left(y<0\right)$;

3) The function $u(x,y)$ satisfies the following condition:

$$u_y(x,-0)=-u_y(x,+0)$$

4) $u(x,y)$ satisfies the following boundary conditions

$$\left\{\delta(s)A_s[u]+\rho(s)u\right\}\Big|_{\sigma}=\varphi(s),\qquad 0<s<l, \tag{3}$$

$$u(x,y)\big|_{AC}=\psi(x),\qquad 0\le x\le\frac{1}{2} \tag{4}$$

where $\delta(s)$, $\rho(s)$, $\varphi(s)$, $\psi(x)$ are given sufficiently smooth functions, and $\psi(x)\in C^1\left[0,\frac{1}{2}\right]\cap C^2\left(0;\frac{1}{2}\right)$, and the consistency condition is fulfilled $\varphi(l)=\psi(0)=0,$ where

$$A_s\left[u\right]=y^m\frac{dy}{ds}\frac{\partial u}{\partial x}-\frac{dx}{ds}\frac{\partial u}{\partial y},$$

$\frac{dx}{ds}=-\cos(n,y),\qquad \frac{dy}{ds}=\cos(n,x)$, where n is the external normal to the curve σ, l is the length of the curve σ, s is the length of an arc of the curve σ, starting from the point $B(1,0)$.

We assume that the curve σ satisfies the following conditions:

1) functions $x(s),\ y(s)$, which describe the parametrical equation of a curve σ, have continuous derivatives $x'(s),y'(s)$, and don't tend to zero at the same time, moreover have the second derivatives meeting Hölder's condition[9] $\kappa\left(0<\kappa<1\right)$ of an order in an interval $0\le s\le l$;

2) in the vicinity end points of the curve σ satisfies inequalities:

$$\left|\frac{dx}{ds}\right|\le C\,y^{m+1}(s), \tag{5}$$

and $x(l)=y(0)=0,\quad x(0)=1,\quad y(l)=0.$ where C is a constant.

The following theorem holds true.

Theorem. *If conditions (2) and*

$$\delta(s)\rho(s)\geq 0, \qquad 0\leq s\leq l \tag{6}$$

$$\lim_{y\to 0}(-y)^{\frac{m}{2}}u^2(1,y)=0 \tag{7}$$

are fulfilled, then the solution of the Problem PT in the domain D is unique.

Proof. We prove the theorem using a method of energy integrals. Let $u(x,y)$ be a twice continuously differentiable solution of (1) in the domain $\bar{D}^{\varepsilon_1,\varepsilon_2}\subset D$, $D^{\varepsilon_1,\varepsilon_2}=D_1^{\varepsilon_1,\varepsilon_2}\cup D_2^{\varepsilon_1,\varepsilon_2}$ where $D_1^{\varepsilon_1,\varepsilon_2}$ is the domain with border $\partial D_1^{\varepsilon_1,\varepsilon_2}=A_{\varepsilon_2}^{\varepsilon_1}B_{\varepsilon_2}^{\varepsilon_1}\cup\sigma_{\varepsilon_1}$ $\left(A_{\varepsilon_2}^{\varepsilon_1}B_{\varepsilon_2}^{\varepsilon_1}:\ y=\varepsilon_2\right)$ strictly lying in the region D_1, and $D_2^{\varepsilon_1,\varepsilon_2}$ is the domain bounded by lines

$$A_{\varepsilon_2}^{\varepsilon_1}B_{\varepsilon_2}^{\varepsilon_1}: y=-\varepsilon_2,\quad A_{\varepsilon_2}^{\varepsilon_1}C_{\varepsilon_1}: x-\frac{2}{m+2}(-y)^{\frac{m+2}{2}}=\varepsilon_1,$$
$$B_{\varepsilon_2}^{\varepsilon_1}C_{\varepsilon_1}: x+\frac{2}{m+2}(-y)^{\frac{m+2}{2}}=1-\varepsilon_1,$$

where $\varepsilon_1,\varepsilon_2$ are sufficiently small positive real numbers.

It is easy to see that in the domain $D_2(y<0)$ the equation (1) has the form:

$$(-y)^m u_{xx}-u_{yy}=0.$$

The following identity holds[10]:

$$u\left[(-y)^m u_{xx}-u_{yy}\right]=\frac{\partial}{\partial x}\left[(-y)^m uu_x\right]-\frac{\partial}{\partial y}\left[uu_y\right]-(-y)^m u_x^2+u_y^2.$$

Integrating the latter over the domain $D_2^{\varepsilon_1,\varepsilon_2}$, we obtain,

$$0=\iint\limits_{D_2^{\varepsilon_1,\varepsilon_2}} u\left[(-y)^m u_{xx}-u_{yy}\right]dxdy=\iint\limits_{D_2^{\varepsilon_1,\varepsilon_2}}\left\{\frac{\partial}{\partial x}\left[(-y)^m uu_x\right]-\frac{\partial}{\partial y}\left[uu_y\right]\right\}dxdy+ + \iint\limits_{D_2^{\varepsilon_1,\varepsilon_2}}\left[u_y^2-(-y)^m u_x^2\right]dxdy \tag{8}$$

Applying the Green's formula (see [12]) to the first integral of the right-hand side of (8), we obtain

$$0=\iint\limits_{D_2^{\varepsilon_1,\varepsilon_2}} u\left[(-y)^m u_{xx}-u_{yy}\right]dxdy=\int\limits_{A_{\varepsilon_2}^{\varepsilon_1}C^{\varepsilon_1}\cup CB_{\varepsilon_2}^{\varepsilon_1}\cup B_{\varepsilon_2}^{\varepsilon_1}A_{\varepsilon_2}^{\varepsilon_1}} u\left[(-y)^m u_x dy+u_y dx\right]+ + \iint\limits_{D_2^{\varepsilon_1,\varepsilon_2}}\left[u_y^2-(-y)^m u_x^2\right]dxdy$$

Calculating the first integral of the right-hand side of the last equality, taking into account the condition on the characteristic AC, we have:

$$\int\limits_{A_{\varepsilon_2}^{\varepsilon_1}C^{\varepsilon_1}\cup CB_{\varepsilon_2}^{\varepsilon_1}\cup B_{\varepsilon_2}^{\varepsilon_1}A_{\varepsilon_2}^{\varepsilon_1}} u\left[(-y)^m u_x dy + u_y dx\right] =$$

$$= \int\limits_{A_{\varepsilon_2}^{\varepsilon_1}C^{\varepsilon_1}} u\left[(-y)^m u_x dy + u_y dx\right] + \int\limits_{CB_{\varepsilon_2}^{\varepsilon_1}} u\left[(-y)^m u_x dy + u_y dx\right] + \int\limits_{B_{\varepsilon_2}^{\varepsilon_1}A_{\varepsilon_2}^{\varepsilon_1}} u\left[(-y)^m u_x dy + u_y dx\right] =$$

$$= \int\limits_{C^{\varepsilon_1}}^{B_{\varepsilon_2}^{\varepsilon_1}} u\left[(-y)^m u_x dy + u_y dx\right] + \int\limits_{1-\varepsilon_1}^{\varepsilon_1} u(x,-\varepsilon_2)u_y(x,\varepsilon_2)dx.$$

Therefore,

$$\int\limits_{C^{\varepsilon_1}}^{B_{\varepsilon_2}^{\varepsilon_1}} u\left[(-y)^m u_x dy + u_y dx\right] + \int\limits_{1-\varepsilon_1}^{\varepsilon_1} u(x,-\varepsilon_2)u_y(x,\varepsilon_2)dx + \iint\limits_{D_2^{\varepsilon_1,\varepsilon_2}} \left[(-y)^m u_x^2 + u_y^2\right]dxdy = 0$$

Furthermore, passing to the limit at $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ considering $u_y(x,-0) = -u_y(x,+0)$, we obtain

$$\int\limits_0^1 \tau(x)\nu(x)dx = -\iint\limits_{D_2}\left[u_y^2 - (-y)^m u_x^2\right]dxdy - \int\limits_C^B u\left[(-y)^m u_x dy + u_y dx\right]$$

On the characteristic BC we have $dx = (-y)^{\frac{m}{2}} dy$. Then

$$\int\limits_C^B u\left[(-y)^m u_x dy + u_y dx\right] = \int\limits_C^B (-y)^{\frac{m}{2}} u\left[u_x dx + u_y dy\right] = \int\limits_C^B (-y)^{\frac{m}{2}} u dy \tag{9}$$

Integrating the last integral by parts, taking into account the conditions $u\big|_{AC} = 0$ and (7), we get:

$$\int\limits_C^B (-y)^{\frac{m}{2}} u du = \frac{m}{4}\int\limits_C^B (-y)^{\frac{m-2}{2}} u^2 dy.$$

By virtue of the condition $-1 < m < 0$ in (9), we have

$$\int\limits_C^B u\left[(-y)^m u_x dy + u_y dx\right] = \int\limits_C^B (-y)^{\frac{m}{2}} u du = \frac{m}{4}\int\limits_C^B (-y)^{\frac{m-2}{2}} u^2 du \le 0\cdot \tag{10}$$

Now we show that the first integral of the right-hand side of (8) is not positive. For this, we pass to the characteristic variables (ξ,η):

$$\xi = x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}}, \quad \eta = x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}}$$

and we get

$$\iint_{D_2}\left[u_y^2 - (-y)^m u_x^2\right]dxdy = -2\iint_{\Delta_1}\left(\frac{m+2}{4}\right)^{\frac{m}{2}}(\eta-\xi)^{\frac{m}{m+2}}u_\xi u_\eta d\xi d\eta, \tag{11}$$

where $\Delta_1 = \{(\xi,\eta): 0 < \xi < 1,\ \xi < \eta < 1\}$ is the image of the domain D_2 in the coordinates (ξ,η).

In the domain Δ_1, equation (1) for $y < 0$ takes the form

$$u_{\xi\eta} - \frac{\beta}{\eta-\xi}(u_\eta - u_\xi) = 0.$$

By multiplying both sides of the last equation by u_η, we have

$$u_\xi u_\eta = u_\eta^2 - \frac{\eta-\xi}{\beta}u_\eta u_{\xi\eta} \tag{12}$$

Substituting (12) into (11)

$$\iint_{D_2}\left[u_y^2 - (-y)^m u_x^2\right]dxdy = -2\left(\frac{m+2}{4}\right)^{\frac{m}{m+2}} \times$$

$$\times\left[\iint_{\Delta_1}(\eta-\xi)^{\frac{m}{m+2}}u_\eta^2 d\xi d\eta - \frac{1}{\beta}\iint_{\Delta_1}(\eta-\xi)^{\frac{2(m+1)}{m+2}}u_\eta u_{\xi\eta}d\xi d\eta\right]$$

and integrating by parts the last integral, we obtain

$$\iint_{D_2}\left[u_y^2 - (-y)^m u_x^2\right]dxdy = \frac{2(m+2)}{m}\left(\frac{m+2}{4}\right)^{\frac{m}{m+2}} \times$$

$$\times\left[\iint_{\Delta_1}(\eta-\xi)^{\frac{m}{m+2}}u_\eta^2 d\xi d\eta + (\eta-\xi)^{\frac{2(m+1)}{m+2}}u_\eta^2\Big|_{\eta=\xi}\right].$$

As the last term on the right-hand side at $\eta = \xi$ vanishes, we have:

$$\iint_{D_2}\left[u_y^2 - (-y)^m u_x^2\right]dxdy = \frac{2(m+2)}{m}\left(\frac{m+2}{4}\right)^{\frac{m}{m+2}}\iint_{\Delta_1}(\eta-\xi)^{\frac{m}{m+2}}u_\eta^2 d\xi d\eta$$

Note that $-1<m<0$ and $\iint\limits_{\Delta_1}(\eta-\xi)^{\frac{m}{m+2}}u_\eta^2 d\xi d\eta \geq 0$, hence we obtain:

$$\iint\limits_{D_2}\left[u_y^2-(-y)^m u_x^2\right]dxdy \leq 0 \qquad (13)$$

Considering (10) and (13) by (8), we get

$$\int\limits_0^1 \tau(x)\nu(x)dx \geq 0 \qquad (14)$$

In the domain D_1 , equation (1) for $y>0$ has the form

$$y^m u_{xx}+u_{yy}=0$$

The following identity is valid

$$u\left[y^m u_{xx}+u_{yy}\right]=\frac{\partial}{\partial x}\left[y^m u u_x\right]+\frac{\partial}{\partial y}\left[u u_y\right]-y^m u_x^2-u_y^2$$

Integrating it over the domain $D_1^{\varepsilon_1,\varepsilon_2}\subset D_1$, we have

$$0=\iint\limits_{D_1^{\varepsilon_1,\varepsilon_2}}u\left[y^m u_{xx}+u_{yy}\right]dxdy=\iint\limits_{D_1^{\varepsilon_1,\varepsilon_2}}\left\{\frac{\partial}{\partial x}\left[y^m u u_x\right]+\frac{\partial}{\partial y}\left[u u_y\right]\right\}dxdy- \\ -\iint\limits_{D_1^{\varepsilon_1,\varepsilon_2}}\left[y^m u_x^2+u_y^2\right]dxdy$$

Applying the Green's formula (see [11,12]) to the first integral of the right-hand side of the last equality, we obtain

$$0=\iint\limits_{D_1^{\varepsilon_1,\varepsilon_2}}u\left[y^m u_{xx}+u_{yy}\right]dxdy=-\iint\limits_{D_1^{\varepsilon_1,\varepsilon_2}}\left[y^m u_x^2+u_y^2\right]dxdy+\int\limits_{\partial D_1^{\varepsilon_1,\varepsilon_2}}u\left[y^m u_x dy-u_y dx\right]$$

By virtue of the condition AB: $y=0\Rightarrow dy=0$ and $dx=-\cos(n,y)ds$, we get

$$0=-\iint\limits_{D_1^{\varepsilon_1,\varepsilon_2}}\left[y^m u_x^2+u_y^2\right]dxdy-\int\limits_{x_1}^{x_2}u(x,\varepsilon_2)u_y(x,\varepsilon_2)dx+\int\limits_{\sigma_{\varepsilon_1}}u\,A_s[u]ds, \qquad (15)$$

where x_1, x_2 are abscissas of points of intersection of the line $y=\varepsilon_2$ with a curve σ_{ε_1}. Taking into account conditions 1) of the Problem PT and $\varphi(s)\equiv b(x)\equiv 0$, taking (3) into account from (15) for $\delta(s)\neq 0$ and $\varepsilon_1\to 0$, $\varepsilon_2\to 0$, we obtain

$$\iint_{D_1}\left[y^m u_x^2+u_y^2\right]dxdy+\int_0^1 \tau(x)\nu(x)dx+\int_\sigma \frac{\delta(s)\rho(s)}{\delta^2(s)}u^2 ds=0. \qquad (16)$$

By virtue of (6) and (14), it follows from (16) that $u_x=u_y=0$ in D_1, that is $u=const$ in $(x,y)\in D_1$, also from the result to the inversion of each term (16) to zero, we also have $u=0$ in $\bar{\sigma}$.

Considering the Hopf's principle, we conclude that $u\equiv 0$ in $\bar{D}_1$ for $\delta(s)\neq 0$. The uniqueness of a solution to the Cauchy problem implies that $u\equiv 0$ and in D_2. Since $\bar{D}=\bar{D}_1\cup\bar{D}_2$ then $u\equiv 0$ in $\bar{D}$. This proves the uniqueness of the solution of Problem PT.

The theorem is proved. 2

Remark. In [5] the uniqueness of the solution of the Problem PT for $\rho(s)\neq 0,\ \forall\ s\in[0,l]$ is proved by the maximum principle.

The existence of a solution of the Problem PT for $\delta(s)\neq 0$ is proved with the help of the method of integral equations[8].

The applicability of the groundwater model to the real situation depends on the accuracy of the input data and parameters. Their determination requires significant research, such as collecting hydrological data (precipitation, evapotranspiration, irrigation, drainage) and determining the parameters mentioned earlier, including pumping. Since many parameters are highly variable in space, expert judgment is required to obtain representative values.

4 Conclusions

Models can also be used for if-then analysis: if the parameter value is A, then what is the outcome, and if the parameter value is B instead, what impact? This analysis may be sufficient to provide a rough idea of groundwater behavior. Still, it can also serve as a sensitivity analysis to answer the question of which factors have a large influence and which have a lesser effect. This information makes it possible to direct investigative efforts toward influential factors. When enough data has been collected, the missing information can be determined using calibration. This means that one assumes a range of values for an unknown or questionable value for a certain parameter and runs the model repeatedly, comparing the results with known corresponding data. For example, suppose groundwater salinity values are available, and the hydraulic conductivity value is uncertain. In that case, a range of conductivity is assumed, and this conductivity value is chosen as "true", which gives salinity results close to the observed values, which means that the groundwater flow, determined by the hydraulic conductivity, follows salinity conditions. This procedure is similar to flow measurement in a river or canal, allowing very salty water of known salt concentration to drip into the canal and measuring the resulting salt concentration downstream.

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