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Deformation and damage capacity of thin–walled rods and tubular conduits under alternating loading

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Abstract. The paper presents deformation and elastoplastic calculation of thin–walled rods (pipelines) under spatial – alternating loading taking into account damageability of material. On the basis of deformation theory and variational principle of Hamilton – Ostrogradsky the system of differential equations of motion (equilibrium) under alternating loading is obtained and the boundary value problems for structural elements are formulated. The algorithms and results of realization of calculation of thin–walled rods (pipes) under alternating loading in view of damage accumulation are given. Numerical results of displacement and force components depending on the number of loading cycles with regard for strain diagrams are given. Effects of secondary plastic deformations and elastic unloading on stress–strain states are shown graphically.

1 Introduction

The functioning of most load–bearing elements in thin–walled structures occur against the background of materials exceeding elasticity limits. When cyclic loads are applied, this leads to a number of additional phenomena, such as the occurrence of secondary plastic deformations, changes in deformation diagrams from cycle to cycle, manifestation of cyclic hardening –softening properties, accumulation of damage leading to material failure. The presence of damage and various types of defects leads to a significant reduction in the strength of rods and pipelines, durability of the structure and contributes to its premature failure. Failures of pipelines lead to high material costs for elimination of accidents and pollution of the environment. Therefore, analysis of causes of damage and destruction of structural elements is of great importance. Investigation of stress – strain state of structural elements with damages causes significant difficulties, since in local areas of stress concentration considerably exceeds yield strength and calculation is performed in elastoplastic domain.

General formulation of phenomenological approach to description of damage accumulation was given by A.A. Ilyushin [1]. The works by V.V. Moskvitinin [2, 3] formulated basic equations of alternating plasticity and viscoplasticity with allowance for accumulated damage and proved theorems about alternating loading and secondary plastic deformations. An effective method of elastic solutions for elastic – plastic problems has

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been proposed by them. A number of convenient modifications of this method currently exist to permit faster convergence of iterative processes. The papers by T. Buriev [4] discuss the implementation and construction of an algorithmic system for calculating load – bearing elements of structures within and beyond elasticity under alternating loads.

In [5] modern problems of estimating resistance of materials and structural elements to a wide range of damaging factors are considered. The application of equations of state, deformation and fracture models under short – and long – term, low – cycle and multi – cycle loading is substantiated. A detailed description of cyclic deformation diagrams for various structural materials is given in [6] and possible clarifications of diagram interpretations in solving cyclic strength problems are shown.

The paper [7] is devoted to the problem of constructing mathematical models of damaged thermoelastically – viscoplastic media and methods for determining "non – standard" model constants related to damage parameters and subjected to experimental determination. The paper [8] deals with computer simulation of deformation, damage and continuum fracture of nonlinear materials and structures.

In [9] equations of equilibrium are derived and methods for solving elastic, elastoplastic and viscoelastoplastic rods are proposed. Numerical research has been done on the mode of action of three – layer rods and plates under single and cyclic loads [10, 11].

In [12, 13], the parameters characterizing nonlinear elastic, elastoplastic and viscoplastic properties of interaction of underground pipeline with the ground are determined on the basis of experimental results. Local laws of interaction of extended underground structures with soils of disturbed and undisturbed structure were constructed [14]. The problem of ensuring low–cycle strength and reliability of trunk pipelines is a multifaceted problem. One of the main directions is development of strength calculations and analysis of pipeline stresses with regard to physical and geometrical nonlinearities [15–18].

It follows from the brief review that the most important problems in this field are to develop deformation models of thin–walled structure elements, to improve the method of calculation and analysis of the mode of deformation under alternating loads taking into account the damaging properties of materials. Calculation models on the basis of deformation theory and Hamilton – Ostrogradsky variation principle and derivation of differential equations of motion (equilibrium) under spatially alternating loading are considered.

2 Objects and methods of research

It follows from the brief review that the most important tasks in this direction are to develop deformation models of thin–walled structure elements, to improve the method of calculation and analysis of the mode of deformation under alternating loads taking into account the damageability of materials. The aim is to develop models on the basis of deformation theory and Hamilton – Ostrogradsky variational principle, to derive system of differential equations of motion (equilibrium) under space – variable loads, to formulate boundary value problems and to apply finite difference method.

2.1 Problem statement and deformation of thin–walled rods (pipelines) by refined theories under initial loading

1) Firstly, the problem statement and the deformation of thin – walled rods (pipelines) according to refined theories under initial loading are given. Let's give the problem statement and the scheme of realization of calculation of thin – walled rods under space – variable loading from the initial state on the basis of small elastic – plastic deformations theory of A.A. Ilyushin [1] and the refined theory of rods proposed by V.Z. Vlasov, G.Yu.

It is well known that under spatial loading, i.e. under combined longitudinal, transverse and torsional forces, the distribution laws of motions, deformations and stresses in rod sections are complex, so the refined theory is based on a number of static hypotheses [20].

Consider a thin – walled rod of arbitrary cross–section under the action of external forces. The *OX* axis is directed along the length of the rod and the *OZ* and *OY* axes are directed along the cross – section. The distribution law of the external load is shown in Fig.1.

The displacements of the centre line of the rod under initial loading are denoted by u_i' , the strain and stress components by e'_{ij}, σ'_{ij}.

The displacements of the points of a rod under the combined action of longitudinal, transverse and torsional forces can be represented in the following form [20]:

$$u_1 = u - y\alpha_1 - z\alpha_2 + \varphi\nu + a_1\beta_1 + a_2\beta_2, \quad u_2 = v - z\theta, \quad u_3 = w + y\theta, \tag{1}$$

where u, v, w – displacement components under first loading; $,\alpha_1\ \alpha_2$ – section angles under pure bending; $,\beta_1\ \beta_2$ – transverse shear angles; θ – torsion angle; ν_1 – shear twist; φ – Saint – Wenans torsion function.

According to the Cauchy formula, taking into account (1), determine the deformation components:

$$\begin{gathered} e_{11} = \frac{\partial u}{\partial x} - y\frac{\partial \alpha_1}{\partial x} - z\frac{\partial \alpha_2}{\partial x} + \varphi(y,z)\frac{\partial \nu}{\partial x} + a_1(y,z)\frac{\partial \beta_1}{\partial x} + a_2(y,z)\frac{\partial \beta_2}{\partial x}; \\ e_{13} = \frac{\partial w}{\partial x} + y\frac{\partial \theta}{\partial x} - \alpha_2^{(n)} + \frac{\partial \varphi}{\partial z}\nu + \frac{\partial a_1}{\partial z}\beta_1 + \frac{\partial a_2}{\partial z}\beta_2, \\ e_{12} = \frac{\partial v}{\partial x} - z\frac{\partial \theta}{\partial x} - \alpha_1 + \frac{\partial \varphi}{\partial y}\nu + \frac{\partial a_1}{\partial y}\beta_1 + \frac{\partial a_2}{\partial y}\beta_2. \end{gathered} \tag{2}$$

According to Ilyushin's theory of small elastic – plastic deformations, the stress components are related through deformation as follows:

$$\sigma_{11} = 3G(1-\omega)e_{11}, \quad \sigma_{13} = G(1-\omega)e_{13}, \quad \sigma_{12} = G(1-\omega)e_{12}; \tag{3}$$

In linear hardening

$$\begin{cases} 0, & at\ \varepsilon_u \le \varepsilon_s \\ \lambda_n\left[1 - \dfrac{\varepsilon_s}{\varepsilon_u}\right] & , at\ \varepsilon_u > \varepsilon_s \end{cases}$$

To derive the equations of motion of rods (pipelines) under spatial loading taking into account elastic – plastic deformations, we use the Hamilton – Ostrogradsky variational principle [20]:

$$\delta\int_t (T - \Pi + A)dt = 0. \tag{4}$$

First determine the variations in kinetic energy, using the ratio

$$\delta\int_t Tdt = \int_t\int_v \rho\sum_{i=1}^{3}\left(\frac{\partial u_i}{\partial t}\cdot\delta\frac{\partial u_i}{\partial t}\right)dvdt. \quad (5)$$

By performing integration operations piecemeal, we obtain

$$\delta\int_t Tdt = \int_V \rho\sum_{i=1}^{3}\left[\frac{\partial u_i}{\partial t}\cdot\delta u_i\right]dv\Big|_t - \int_t\int_V \rho\sum_{i=1}^{3}\left[\frac{\partial^2 u_i}{\partial t^2}\cdot\delta u_i\right]dvdt. \quad (6)$$

In relation (6) we denote the first and second terms by I_1, I_2 and rewrite them as follows:

$$I_1 = \int_V \rho\left[\frac{\partial u_1}{\partial t}\cdot\delta u_1 + \frac{\partial u_2}{\partial t}\delta u_2 + \frac{\partial u_3}{\partial t}\delta u_3\right]dV\Big|_t \quad (7)$$

$$I_2 = \int_t\int_V \rho\left[\frac{\partial^2 u_1}{\partial t^2}\cdot\delta u_1 + \frac{\partial^2 u_2}{\partial t^2}\delta u_2 + \frac{\partial^2 u_3}{\partial t^2}\delta u_3\right]dVdt. \quad (8)$$

Now substitute the displacement expressions (1) with the variation signs (7):

$$I_1 = \int_V \rho\left[\frac{\partial u_1}{\partial t}\cdot\delta(u - y\alpha_1 - z\alpha_2 + \varphi v + a_1\beta_1 + a_2\beta_2) + \right.$$
$$\left. + \frac{\partial u_2}{\partial t}\delta(v - z\theta) + \frac{\partial u_3}{\partial t}\delta(w + \theta)\right]dV\Big|_t .$$

Open the brackets and perform integration operations on the bar section, introducing the following notations:

$$\int_F dF = F_x,\quad \int_F y\varphi dF = S_\varphi,\quad \int_F ydF = S_z,\quad \int_F zdF = S_y,\quad \int_F a_1 dF = S_{a_1},\quad \int_F a_2 dF = S_{a_2},$$
$$\int_F y^2 dF = J_z,\quad \int_F z^2 dF = J_y,\quad \int_F (y^2+z^2)dF = J_\rho,\quad \int_F yzdF = J_{yz},\quad \int_F \varphi^2 dF = J_\varphi,$$
$$\int_F a_1\varphi\, dF = J_{a_1\varphi},\quad \int_F a_2\varphi\, dF = J_{a_2\varphi},\quad \int_F za_1\, dF = J_{za_1},\quad \int_F za_2 dF = J_{za_2},\quad \int_F ya_1\, dF = J_{ya_1},$$
$$\int_F y\varphi dF = J_{y\varphi},\quad \int_F z\varphi dF = J_{z\varphi},\quad \int_F a_1^2 dF = J_{a_1},\quad \int_F a_2^2 dF = J_{a_2},$$
$$\int_F a_1 a_2 dF = J_{a_1 a_2},\quad \int_F ya_2 dF = J_{ya_2}.$$

Let's write the integral I_1 with regard to the notations as follows:

$$I_1 = \int_x \Bigg\{ \left[F\frac{\partial u}{\partial t} - S_z\frac{\partial \alpha_1}{\partial t} - S_y\frac{\partial \alpha_2}{\partial t} + S_\varphi\frac{\partial \nu}{\partial t} + + S_{a_1}\frac{\partial \beta_1}{\partial t} + S_{a_2}\frac{\partial \beta_2}{\partial t} \right]\delta u + \left[F\frac{\partial \upsilon}{\partial t} - \right.$$
$$\left. - S_y\frac{\partial \theta}{\partial t} \right]\delta\upsilon + \left[F\frac{\partial w}{\partial t} + S_z\frac{\partial \theta}{\partial t} \right]\delta w - \left[S_z\frac{\partial u}{\partial t} - J_z\frac{\partial \alpha_1}{\partial t} - J_{yz}\frac{\partial \alpha_2}{\partial t} + J_{y\varphi}\frac{\partial \nu}{\partial t} + J_{ya_1}\frac{\partial \beta_1}{\partial t} + \right.$$
$$\left. + J_{ya_2}\frac{\partial \beta_2}{\partial t} \right]\delta\alpha_1 - \left[S_y\frac{\partial u}{\partial t} - J_{yz}\frac{\partial \alpha_1}{\partial t} - J_y\frac{\partial \alpha_2}{\partial t} + J_{z\varphi}\frac{\partial \nu}{\partial t} + J_{za_1}\frac{\partial \beta_1}{\partial t} + J_{za_2}\frac{\partial \beta_2}{\partial t} \right]\delta\alpha_2 + \left[S_\varphi\frac{\partial u}{\partial t} - \right.$$
$$\left. - J_{y\varphi}\frac{\partial \alpha_1}{\partial t} - J_{z\varphi}\frac{\partial \alpha_2}{\partial t} + J_\varphi\frac{\partial \nu}{\partial t} + J_{\varphi a_1}\frac{\partial \beta_1}{\partial t} + J_{\varphi a_2}\frac{\partial \beta_2}{\partial t} \right]\delta\nu + \left[S_z\frac{\partial w}{\partial t} - - S_y\frac{\partial \upsilon}{\partial t} + J_\rho\frac{\partial \theta}{\partial t} \right]\delta\theta +$$
$$+ \left[S_{a_1}\frac{\partial u}{\partial t} - J_{ya_1}\frac{\partial \alpha_1}{\partial t} - J_{za_1}\frac{\partial \alpha_2}{\partial t} + J_{\varphi a_1}\frac{\partial \nu}{\partial t} + J_{a_1}\frac{\partial \beta_1}{\partial t} + J_{a_1 a_2}\frac{\partial \beta_2}{\partial t} \right]\delta\beta_1 +$$
$$+ \left[S_{a_2}\frac{\partial u}{\partial t} - J_{ya_2}\frac{\partial \alpha_1}{\partial t} - J_{za_2}\frac{\partial \alpha_2}{\partial t} + J_{\varphi a_2}\frac{\partial \nu}{\partial t} + J_{a_1 a_2}\frac{\partial \beta_1}{\partial t} + J_{a_2}\frac{\partial \beta_2}{\partial t} \right]\delta\beta_2 \Bigg\} dx \Bigg|_t .$$

Similarly determine the second part of the kinetic equation (8), that is, the expression of the integral I_2

$$I_2 = \int_t\int_x \Bigg\{ \left[F\frac{\partial^2 u}{\partial t^2} - S_z\frac{\partial^2 \alpha_1}{\partial t^2} - S_y\frac{\partial^2 \alpha_2}{\partial t^2} + S_\varphi\frac{\partial^2 \nu}{\partial t^2} + S_{a_1}\frac{\partial^2 \beta_1}{\partial t^2} + S_{a_2}\frac{\partial^2 \beta_2}{\partial t^2} \right]\delta u - \left[S_z\frac{\partial^2 u}{\partial t^2} - J_z\frac{\partial^2 \alpha_1}{\partial t^2} - \right.$$
$$\left. - J_{yz}\frac{\partial^2 \alpha_2}{\partial t^2} + J_{y\varphi}\frac{\partial^2 \nu}{\partial t^2} + J_{ya_1}\frac{\partial^2 \beta_1}{\partial t^2} + J_{ya_2}\frac{\partial^2 \beta_2}{\partial t^2} \right]\delta\alpha_1 - \left[S_y\frac{\partial^2 u}{\partial t^2} - J_{yz}\frac{\partial^2 \alpha_1}{\partial t^2} - J_y\frac{\partial^2 \alpha_2}{\partial t^2} + J_{z\varphi}\frac{\partial^2 \nu}{\partial t^2} + \right.$$
$$\left. + J_{za_1}\frac{\partial^2 \beta_1}{\partial t^2} + J_{za_2}\frac{\partial^2 \beta_2}{\partial t^2} \right]\delta\alpha_2 + \left[S_\varphi\frac{\partial^2 u}{\partial t^2} - J_{\varphi z}\frac{\partial^2 \alpha_1}{\partial t^2} - J_{z\varphi}\frac{\partial^2 \alpha_2}{\partial t^2} + J_\varphi\frac{\partial^2 \nu}{\partial t^2} + J_{\varphi a_1}\frac{\partial^2 \beta_1}{\partial t^2} + J_{\varphi a_2}\frac{\partial^2 \beta_2}{\partial t^2} \right]\delta\nu +$$
$$+ \left[S_{a_1}\frac{\partial^2 u}{\partial t^2} - J_{ya_1}\frac{\partial^2 \alpha_1}{\partial t^2} - J_{za_1}\frac{\partial^2 \alpha_2}{\partial t^2} + J_{a_1\varphi}\frac{\partial^2 \nu}{\partial t^2} + J_{a_1}\frac{\partial^2 \beta_1}{\partial t^2} + J_{a_1 a_2}\frac{\partial^2 \beta_2}{\partial t^2} \right]\delta\beta_1 + \left[S_{a_2}\frac{\partial^2 u}{\partial t^2} - J_{ya_2}\frac{\partial^2 \alpha_1}{\partial t^2} - \right.$$
$$\left. - J_{za_2}\frac{\partial^2 \alpha_2}{\partial t^2} + J_{\varphi a_2}\frac{\partial^2 \nu}{\partial t^2} + J_{a_1 a_2}\frac{\partial^2 \beta_1}{\partial t^2} + J_{a_2}\frac{\partial^2 \beta_2}{\partial t^2} \right]\delta\beta_2 + \left[S_z\frac{\partial^2 w}{\partial t^2} - S_y\frac{\partial^2 \upsilon}{\partial t^2} + J_\rho\frac{\partial^2 \theta}{\partial t^2} \right]\delta\theta +$$
$$+ \left[F\frac{\partial^2 \upsilon}{\partial t^2} - S_y\frac{\partial^2 \theta}{\partial t^2} \right]\delta\upsilon + \left[F\frac{\partial^2 w}{\partial t^2} + S_z\frac{\partial^2 \theta}{\partial t^2} \right]\delta w \Bigg\} dxdt.$$

Given the expressions of the integrals I_1 and I_2, of the variation of kinetic energy (6) we write in vector form

$$\delta\int_t T dt = \int_x \widetilde{A}\frac{\partial Y}{\partial t} E\delta Y dx\Big|_t - \int_t\int_x \widetilde{A}\frac{\partial^2 Y}{\partial t^2} E\delta Y \; dxdt, \tag{9}$$

where $Y = \{u, \upsilon, w, \alpha_1, \alpha_2, \theta, \nu, \beta_1, \beta_2\}$ is a displacement vector; $\widetilde{A} -$ is a ninth–order matrix; E is a unit matrix.

The variation of the potential energy of the rod in this formulation is

$$\delta\int_t \Pi dt = \int_t\int_v\left(\sum_{i=1}^{3}\sigma_{i1}\delta e_{i1}\right)dvdt = \int_t\int_V\left[\sigma_{11}\delta e_{11} + \sigma_{12}\delta e_{12} + \sigma_{13}\delta e_{13}\right]dVdt. \quad (10)$$

Let's substitute the deformation expression (2) into (13):

$$\delta\int_t \Pi dt = \int_t\int_V\left\{\sigma_{11}\delta\left(\frac{\partial u}{\partial x} - y\frac{\partial\alpha_1}{\partial x} - z\frac{\partial\alpha_2}{\partial x} + \varphi\frac{\partial\nu}{\partial x} + a_1\frac{\partial\beta_1}{\partial x} + a_2\frac{\partial\beta_2}{\partial x}\right) + \right.$$
$$+\sigma_{13}\delta\left(\frac{\partial w}{\partial x} + y\frac{\partial\theta}{\partial x} - \alpha_2 + \frac{\partial\varphi}{\partial z}\nu + \frac{\partial a_1}{\partial z}\beta_1 + \frac{\partial a_2}{\partial z}\beta_2\right) +$$
$$\left. +\sigma_{12}\delta\left[\frac{\partial\upsilon}{\partial x} - z\frac{\partial\theta}{\partial x} - \alpha_1 + \frac{\partial\varphi}{\partial y}\nu + \frac{\partial\alpha_1}{\partial y}\beta_1 + \frac{\partial\alpha_2}{\partial y}\beta_2\right]\right\}dVdt. \quad (11)$$

Let us transform the variation of potential energy (11). To do this, open the brackets under the variation sign and select the integral over the rod cross section. After some calculations and notations from (11) we have:

$$\delta\int_t \Pi dt = \int_t\left\{N_x\delta u - M_z\delta\alpha_1 - M_y\delta\alpha_2 + Q_y\delta\upsilon + Q_z\delta w + M_x\delta\theta + \right.$$
$$\left. + M_z\delta\nu + M_{\alpha_1}\delta\beta_1 + M_{\alpha_2}\delta\beta_2\right\}dt\Big|_x - \int_t\int_x\left\{\frac{\partial N_x}{\partial x}\delta u + \frac{\partial Q_y}{\partial x}\delta\upsilon + \frac{\partial Q_z}{\partial x}\delta w + \right.$$
$$+\left(Q_y - \frac{\partial M_z}{\partial x}\right)\delta\alpha_1 + \left(Q_z - \frac{\partial M_y}{\partial x}\right)\delta\alpha_2 + \frac{\partial M_x}{\partial x}\delta\theta + \left(\frac{\partial M_\varphi}{\partial x} - Q_\nu\right)\delta\nu +$$
$$\left. +\left(\frac{\partial M_{a_1}}{\partial x} - Q_{\beta_1}\right)\delta\beta_1 + \left(\frac{\partial M_{a_2}}{\partial x} - Q_{\beta_2}\right)\delta\beta_2\right\}dxdt. \quad (12)$$

The following notations are introduced here:

$$\int_F\sigma_{11}dF = N_x,\quad \int_F\sigma_{12}dF = Q_y,\quad \int_F\sigma_{11}ydF = M_z,\quad \int_F\varphi\sigma_{11}dF = M_\varphi\quad \int_F a_1\sigma_{11}dF = M_{a_1},\quad \int_F a_2\sigma_{11}dF = M_{a_2},$$
$$\int_F(\sigma_{13}y - \sigma_{12}z)dF = M_x,\quad \int_F\sigma_{11}zdF = M_y,\quad \int_F\sigma_{13}dF = Q_z,\quad \int_F\left(\sigma_{13}\frac{\partial\varphi}{\partial z} + \sigma_{12}\frac{\partial\varphi}{\partial y}\right)dF = Q_\nu,$$
$$\int_F\sigma_{13}\frac{\partial a_1}{\partial z}dF = Q_{\beta_1},\quad \int_F\sigma_{13}\frac{\partial a_2}{\partial z}dF = Q_{\beta_2}. \quad (13)$$

Given relations (3) and notation (13), the expressions for the internal forces and moments, e.g. N_x and M_{a_2} , can be represented as follows

$$N_x = 3G\left\{\left(\tilde{F} - \tilde{F}_\omega\right)\frac{\partial u}{\partial x} - \left(S_z - S_{z\omega}\right)\frac{\partial \alpha_1}{\partial x} - \left(S_y - S_{y\omega}\right)\frac{\partial \alpha_2}{\partial x} + \right.$$

$$\left. + \left(S_\varphi - S_{\varphi\omega}\right)\frac{\partial \nu_1}{\partial x} + \left(S_{a_1} - S_{a_1\omega}\right)\frac{\partial \beta_1}{\partial x} + \left(S_{a_2} - S_{a_2\omega}\right)\frac{\partial \beta_2}{\partial x}\right\}; \qquad (14)$$

$$M_{a_2} = 3G\left\{\left(S_{a_2} - S_{a_2\omega}\right)\frac{\partial u}{\partial x} + \left(I_{a_2 y} - I^{\omega}_{a_2 y}\right)\frac{\partial \alpha_1}{\partial x} + \left(I_{a_1 z} - I^{\omega}_{a_1 z}\right)\frac{\partial \alpha_2}{\partial x} + \right.$$

$$\left. + \left(I_{a_2\varphi} - I^{\omega}_{a_2\varphi}\right)\frac{\partial \nu_1}{\partial x} + \left(I_{a_1 a_2} - I^{\omega}_{a_1 a_2}\right)\frac{\partial \beta_1}{\partial x} + \left(I_{a_2} - I_{a_2\omega}\right)\frac{\partial \beta_2}{\partial x}\right\}.$$

where

$$F_x = \int_F dF, \quad S_z = \int_F y\,dF, \quad S_y = \int_F z\,dF, \quad S_\varphi = \int_F \varphi dF, \quad S_{a_1} = \int_F a_1 dF, \quad S_{a_2} = \int_F a_2 dF,$$

$$J_{a_2 y} = \int_F a_2 y dF, \quad J_{a_2 z} = \int_F a_2 z dF, J_{a_2\varphi} = \int_F \varphi a_2\, dF, \quad J_{a_1 a_2} = \int_F a_1 a_2 dF, \quad J_{a_2} = \int_F a_2^2 dF.$$

Integrals $F^{)}_{\omega}, \ldots, J^{\omega}_{a_2}$ containing plasticity functions ω, e.g.
$F_{\omega} = \int_F \omega dF, \ldots, J^{\omega}_{a_2} = \int_F \omega a_2^2 dF$ etc. are defined in a similar way.

Let's substitute expressions of internal forces and moments (14) into variations of potential energy (12). Introducing some notations the variations of potential energy will be presented in vector form

$$\delta\int_t \Pi dt = \int_t \left\{\left(A^{yn} - A^{n\pi}\right)\frac{\partial Y}{\partial x} + \left(B^{yn} - B^{n\pi}\right)Y\right\}E\delta Y dt\Big|_x + \int_t\int_x \left\{\frac{\partial}{\partial x}\left(\left(A^{yn} - A^{n\pi}\right)\frac{\partial Y}{\partial x} + \right.\right.$$

$$\left.\left. + \left(B^{yn} - B^{n\pi}\right)Y\right) + \left(C^{yn} - C^{n\pi}\right)\frac{\partial Y}{\partial x} + \left(D^{yn} - D^{n\pi}\right)Y\right\}E\delta Y dx dt\ . \qquad (15)$$

Consider variations in the work of external forces:

$$\delta\int_t A dt = \int_t\int_V \sum_{i=1}^{3} p_i \delta u_i d\upsilon dt + \int_t\int_s \sum_{i=1}^{3} q_i \delta u_i ds dt + \int_t\int_{s_1} \sum_{i=1}^{3} f_i \delta u_i ds_1 dt\Big|_x, \qquad (16)$$

where p_i – bulk forces at initial loading; q_i – surface forces; f_i – end forces.

To the relation (16) we substitute the displacement expressions (1). Open the brackets and integrate over the cross – sections of the rod:

$$\delta\int_t Adt = \int_t\int_x\left\{\int_F [p_1\delta(u - y\alpha_1 - z\alpha_2 + \varphi\nu + a_1\beta_1 + a_2\beta_2) + p_2\delta(\nu - z\theta) + \right.$$
$$+ p_3\delta(w + y\theta)]dF\}dxdt + \int_t\int_x\left\{\int_l [q_1\delta(u - y\alpha_1 - z\alpha_2 + \varphi\nu + a_1\beta_1 + a_2\beta_2) + \right.$$
$$+ q_2\delta(\nu - z\theta) + q_3\delta(w + y\theta)]dl\}dxdt + \int_t\left\{\int_{s_1} [f_1\delta(u - y\alpha_1 - z\alpha_2 + \varphi\nu + \right.$$
$$+ a_1\beta_1 + a_2\beta_2) + f_1\delta(\nu_2 - z\theta) + f_3\delta(w + y\theta)]dS_1\}dt\Big|_x. \qquad (17)$$

Let's introduce notations:

$$\int_F p_1 dF = N_x^{об}, \quad \int_F yp_1 dF = M_z^{об}, \quad \int_F zp_1 dF = M_y^{об)},$$
$$\int_F \varphi\, p_1 dF = M_\varphi^{об}, \quad \int_F a_1 p_1 dF = M_{a_1}^{об}, \quad \int_F a_2 p_1 dF = M_{a_2}^{об},$$
$$\int_F p_3 dF = Q_y^{об}, \quad \int_F p_3 dF = Q_z^{об}, \quad \int_F (yp_3 - zp_2)dF = M_x^{об}.$$

Similarly, surface and end loads are defined, e.g. $N_x^{П}$, $N_x^{Г}$. Let us now rewrite relations (17) as

$$\delta\int_t Adt = \int_t\int_x\{(N^{об} - N^{П})\delta u + (Q_y^{об} + Q_y^{П})\delta\upsilon + (Q_z^{об} + Q_z^{П})\delta w - (M_z^{об} + M_z^{П})\delta\alpha_1 +$$
$$-(M_y^{об} + M_y^{П})\delta\alpha_2 - (M_x^{об} + M_x^{П})\delta\theta + (M_\varphi^{об} + M_\varphi^{П})\delta\nu + (M_{a_1}^{об} + M_{a_1}^{П})\delta\beta_1 -$$
$$+(M_{a_2}^{об} - M_{a_2}^{П})\delta\beta_2\}dxdt + \int_t\{N^{Г}\delta u + Q_y^{Г}\delta\upsilon + Q_z^{Г}\delta w - M_z^{Г}\delta\alpha_1 - M_y^{Г(n)}\delta\alpha_2^{(n)} -$$
$$- M_x^{Г(n)}\delta\theta^{(n)} + M_\varphi^{Г(n)}\delta\nu^{(n)} + M_{a_1}^{об(n)}\delta\beta_1^{(n)} + M_{a_2}^{об(n)}\delta\beta_2^{(n)}\}dt\Big|_x. \qquad (18)$$

In the case of considering the force of interaction with the medium $q_i^{(n)}$ – surface forces; $f_i^{(n)}$ – torsional forces, taken as follows [12,13]:

$$q_i = -k_i^{(П)}(u_i - u_i^0) + \tilde{q}_i, \quad f_i = -k_i^{(T)}(u_i - u_i^0) + \tilde{f}_i,$$

where $k_i^{(П)}$ – coefficients of interaction of the rod with the environment at the surface of the rod; $k_i^{(T)}$ – coefficients of interaction of the rod with the environment at the ends; u_i^0 – components of the spatial seismic movement of the ground along the coordinate axes and determined by analogy (1):

$$u_1^0 = u^0 - y\alpha_1^0 - z\alpha_2^0 + \varphi v^0 + a_1\beta_1^0 + a_2\beta_2^0,$$
$$u_2^0 = v^0 - z\theta^0, \quad u_3^0 = w^0 + y\theta^0. \tag{19}$$

According to (1) and (19), the expressions q_i, f_i will take the form, for example,

$$q_1 = -k_1\left[(u-u^0) - y(\alpha_1 - \alpha_1^0) - z(\alpha_2 - \alpha_2^0) + \varphi(v - v^0) + a_1(\beta_1 - \beta_1^{0)}) + \right.$$
$$\left. + a_2(\beta_2 - \beta_2^{0)}) + \tilde{q}_1\right], \quad q_2 = -k_2\left[(v-v^0) - z(\theta - \theta^{0)}) + \tilde{q}_2\right],$$
$$q_3 = -k_3\left[(w-w^0) + y(\theta - \theta^0) + \tilde{q}_3\right].$$

In the same way, we define f_1, f_2, f_3. Substitute the expressions q_i, f_i in the variations of the work of external forces and, introducing a number of notations, we obtain relations of the form (18) [13] with $N_x^{вз}, Q_y^{вз}, \ldots M_{a_2}^{вз}$ addition, i.e. expressions before the variations $\delta u, \ldots, \delta\beta_2$ (18) have the following form: $\left(N_x^{об} + N_x^{П} + N_x^{вз}\right), \ldots, \left(M_{a_2}^{об} + M_{a_2}^{П} + M_{a_2}^{вз}\right)$.

The interaction law and non–linear coefficients determined experimentally with regard to the accumulation of damage in the ground should be further clarified.

Taking into account the introduced notations, the variations of the work of external forces will be presented in vector form:

$$\delta\int_t A dt = \int_t Q^{Г}\delta Y dt\Big|_x + \int_t\int_x Q^{П}\, dY dx dt\,. \tag{20}$$

Substituting the vector expressions of variation of kinetic (9), potential (15) energies and work of external forces (20) into the variational principle (4), we obtain

$$\int_t\int_x\left\{\tilde{A}\frac{\partial^2 Y}{\partial t^2} + \frac{\partial}{\partial x}\left[\left(A^{уп} - A^{пл}\right)\frac{\partial Y}{\partial x} + \left(B^{уп} - B^{пл}\right)Y\right] + \right.$$
$$\left. + \left(C^{уп} - C^{пл}\right)\frac{\partial Y}{\partial x} + \left(D^{уп} - D^{пл}\right)Y + Q^{П}\right\}E\,\delta Y\,dx\,dt +$$
$$+ \int_t\left\{\left(A^{уп} - A^{пл}\right)\frac{\partial Y}{\partial x} + \left(B^{уп} - B^{пл}\right)Y + Q^{Г}\right\}E\,\delta Y\;dt\Big|_x + \int_x \tilde{A}\frac{dY}{dt}E\delta Y\,dx\Big|_t = 0.$$

From this variational equation we obtain the following boundary value problem:

$$\tilde{A}\frac{\partial^2 Y}{\partial t^2} + \frac{\partial}{\partial x}\left[\left(A^{уп} - A^{пл}\right)\frac{\partial Y}{\partial x} + \left(B^{уп} - B^{пл}\right)Y\right] + \left(C^{уп} - C^{пл}\right)\frac{\partial Y}{\partial x} + \left(D^{уп} - D^{пл}\right)Y + Q^{н} = 0; \tag{21}$$

$$\left\{\left(A^{уп} - A^{пл}\right)\frac{\partial Y}{\partial x} + \left(B^{уп} - B^{пл}\right)Y + Q^{гр}\right\}\delta Y\Big|_x = 0;\; \tilde{A}\frac{dY}{dt}E\delta Y\Big|_t = 0. \tag{22}$$

Here the quadratic matrices of order nine *A, B, C, D*, the vectors of external forces of order nine $Q^{н}$ and $Q^{гр}$ and the coefficients have the following form:

$$(a_{ij} = a_{ij}^{уп} - a_{ij}^{пл},\ b_{ij} = b_{ij}^{уп} - b_{ij}^{пл},\ c_{ij} = -b_{ij},\ d_{ij} = d_{ij}^{уп} - d_{ij}^{пл}).$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & 0 & 0 & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & 0 & 0 & 0 \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{77} & a_{78} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{87} & a_{88} & a_{89} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{98} & a_{99} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{72} & 0 & b_{74} & b_{75} & 0 & 0 & 0 & 0 \\ 0 & b_{82} & b_{83} & b_{84} & b_{85} & b_{86} & 0 & 0 & 0 \\ 0 & 0 & b_{93} & b_{94} & 0 & b_{96} & 0 & 0 & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_{22} & 0 & d_{24} & d_{25} & 0 & 0 & 0 & 0 \\ 0 & 0 & d_{33} & d_{34} & d_{35} & 0 & 0 & 0 & 0 \\ 0 & d_{42} & d_{43} & d_{44} & d_{45} & d_{46} & 0 & 0 & 0 \\ 0 & d_{52} & 0 & d_{54} & d_{55} & 0 & 0 & 0 & 0 \\ 0 & 0 & d_{63} & d_{64} & 0 & d_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_{27} & c_{28} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{38} & c_{39} \\ 0 & 0 & 0 & 0 & 0 & 0 & c_{47} & c_{48} & c_{49} \\ 0 & 0 & 0 & 0 & 0 & 0 & c_{57} & c_{58} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{68} & c_{69} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where, for instance

$$a_{11} = \frac{l^2}{I_0}(F - F_\omega);\ a_{12} = \frac{l}{I_0}(S_y - S_{y\omega});\ a_{13} = \frac{l}{I_0}(S_z - S_{z\omega});\ a_{14} = \frac{l}{I_0 h_0}(I^\varphi - I_\omega^\varphi);$$

$$a_{15} = \frac{l}{I_0}(S_y^{a_1} - S_{y\omega}^{a_1});\ a_{16} = \frac{l}{I_0}(S_z^{a_2} - S_{z\omega}^{a_2});\ a_{22} = \frac{1}{I_0}(I_y - I_{y\omega});\ a_{23} = \frac{1}{I_0}(I^{yz} - I_\omega^{yz});$$

$$a_{24} = \frac{1}{I_0 h_0}(I_{yz} - I_{yz\omega});\ a_{25} = \frac{1}{I_0}(I^{a_1 z} - I_\omega^{a_1 z});\ a_{26} = \frac{-1}{I_0}(I^{a_2 z} + I_\omega^{a_2 z}),\ a_{33} = \frac{1}{I_0}(I_z - I_{z\omega});$$

$$a_{34} = -\frac{1}{I_0 h_0}(I^{\varphi y} - I_\omega^{\varphi y)});\ a_{35} = -\frac{1}{I_0}(I^{a_1 y} - I_\omega^{a_1 y});\ a_{36} = \frac{-1}{I_0}(I^{a_2 y} + I_\omega^{a_2 y});$$

$$a_{44} = \frac{1}{I_0 h_0^2}(I^{\varphi^2} - I_\omega^{\varphi^2});\ a_{45} = \frac{1}{I_0 h_0}(I^{a_1\varphi} - I_\omega^{a_1\varphi});\ a_{46} = \frac{1}{I_0 h_0}(I^{a_2\varphi} - I_\omega^{a_2\varphi});$$

$$a_{55} = \frac{1}{I_0}(I^{a_1{}^2} - I_\omega^{a_1 2});\ a_{56} = \frac{1}{I_0}(I^{a_1 a_2} + I_\omega^{a_1 a_2});\ a_{66} = \frac{1}{I_0}(I^{a_2{}^2} - I_\omega^{a_2 2)});$$

$$a_{77} = \frac{l^2}{3I_0}(F^{a_1^1} - F_{\omega}^{a_1^1}), a_{78} = -\frac{l^2}{3I_0 h_0}(S_z^{a_2^1} - S_{z\omega}^{a_2^1}),$$

$$a_{88} = \frac{l^2}{3I_0 h_0^2}\left[(I_y + I_z) - (I_{y\omega} - I_{z\omega})\right];$$

$$a_{89} = \frac{l^2}{3I_0 h_0}(S_y - S_{y\omega}); \;\; a_{99} = \frac{l^2}{3I_0}(F - F_{\omega}).$$

Given (14) from the variation equation (21), we have the following system of differential equations of rod equilibrium at alternating loads with boundary conditions in vector form:

$$\frac{\partial}{\partial x}\left[\left(A^{уп} - A^{пл}\right)\frac{\partial Y}{\partial x} + \left(B^{уп} - B^{пл}\right)Y\right] + \left(C^{уп} - C^{пл}\right)\frac{\partial Y}{\partial x} + \left(D^{уп} - D^{пл}\right)Y = Q^{(П)}. \quad (23)$$

Boundary conditions:

$$\left\{\left(A^{уп} - A^{пл}\right)\frac{\partial Y}{\partial x} + \left(B^{уп} - B^{пл}\right)Y - B^{пло}Y - Q^{(Г)}\right\}\delta Y\bigg|_{Г} = 0 \,. \quad (24)$$

Here Y is the desired ninth – order function. Expressions of internal forces and moments in vector form can be represented as

$$P = \frac{3Gh_0 I_0}{l^3}\left\{\left(\widetilde{A}^{уп} - \widetilde{A}^{пл}\right)\frac{\partial Y}{\partial x} + \left(\widetilde{B}^{уп} - \widetilde{B}^{пл}\right)Y\right\}, \quad (25)$$

where P is a twelfth – order function vector

$$P = \left\{N_x, M_y, M_z, M_{\varphi}, M_{a_1}, M_{a_2}, Q_1, M_x, Q_2, Q_{\bar{a}_1}, Q_{\bar{a}_2}, M_{\bar{\varphi}}\right\}.$$

The matrices $\widetilde{A}^{уп}, \widetilde{A}^{пл}, \widetilde{B}^{уп}, \widetilde{B}^{пл}$ of the twelfth–order square matrix and the elements are described as follows:

$$\widetilde{a}_{ij} = a_{ij}, \quad \widetilde{a}_{10,s} = b_{s,5}; \quad \widetilde{a}_{11,s} = b_{s,6}; \quad \widetilde{a}_{12,s} = b_{s,4}; \;\; \widetilde{b}_{ij} = b_{ij},$$
$$\widetilde{b}_{10,r} = d_{2,s}; \; \widetilde{b}_{11,r} = d_{r,6}; \; \widetilde{b}_{12,r} = d_{r,4}; \; (i,j = 1,2,...9; s = 7,8,9; r = 2,3,4,5,6;) \,.$$

The finite difference method and the Ilyushin elastic solution method are used to solve the boundary value problem. In the process of their approximation a central difference scheme of the second order of accuracy is used [21,22]. The application diagram for the external load distribution is shown in fig. 1.

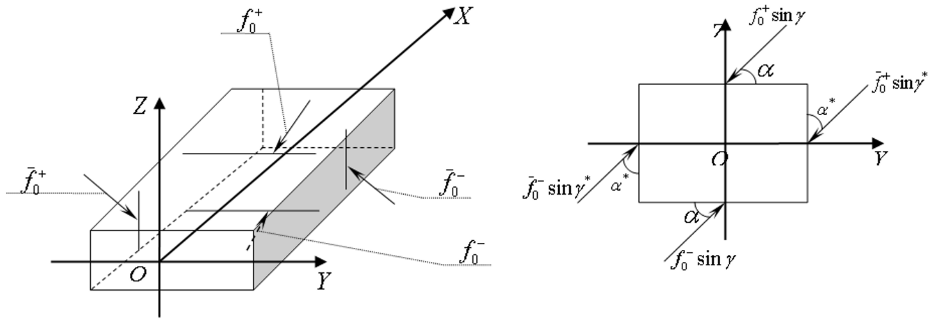

Fig 1. Diagram of external load distribution application

Vector equation (23) after applying the differences of the scheme takes the form:

$$\left(A_i^{уп} - A_i^{пл}\right)Y_{i+1} - \left(B_i^{уп} - B_i^{пл}\right)Y_i + \left(C_i^{уп} - C_i^{пл}\right)Y_i = \widetilde{Q}_i. \tag{26}$$

To solve the formulated algebraic equations (26) with appropriate boundary conditions, the Godunov's run method [4, 21, 23] is used, using the following recurrence formula:

$$V_i = \alpha_i V_{i+1} + \beta_i;\ i = N-1,...,1 \tag{27}$$

Here $\alpha_i = (B_i - C_i\alpha_{i-1})^{-1}A_i;\ \beta_i = (B_i - C_i\alpha_{i-1})^{-1}(C_i\beta_{i-1} - F);\ i = 1,2,...,N-1.$

In order to implement the above algorithm, a modified complex program in an object–oriented language has been written [23].

2) Construction of methodology for problem solution under spatially variable loading of rods (pipes) with allowance for damage accumulation. Let us now consider the construction of a solution for any spatially – variable *n* – th loading of an elastic – plastic rod (pipe) taking into account the accumulation of damageability. Suppose that after *(m–1)* semicycle of loading, starting from the moment t_{m-1}, the instantaneous unloading takes place and a new loading is performed by the forces of the opposite sign. These forces will create the displacement field $u_i^{(n)}$, deformation $\varepsilon_{ij}^{(n)}$ and stresses $\sigma_{ij}^{(n)}$.

Following the theory of V.V. Moskvitin [2], let us introduce differences of the following kind

$$\begin{gathered}\overline{u}_i^{(n)} = (-1)^n(u_i^{(n-1)} - u_i^{(n)}),\ \overline{e}_{ij}^{(n)} = (-1)^n(e_{ij}^{(n-1)} - e_{ij}^{(n)}),\\ \overline{\sigma}_{ij}^{(n)} = (-1)^n(\sigma_{ij}^{(n-1)} - \sigma_{ij}^{(n)}).\end{gathered} \tag{28}$$

The displacements of the points of the rod under alternating loading, by analogy with (1), will be represented as

$$\begin{gathered}\overline{u}_1^{(n)} = \overline{u}^{(n)} - y\overline{\alpha}_1^{(n)} - z\overline{\alpha}_2^{(n)} + \varphi\overline{v}^{(n)} + a_1\overline{\beta}_1^{(n)} + a_2\overline{\beta}_2^{(n)},\\ \overline{u}_2^{(n)} = \overline{v}^{(n)} - z\overline{\theta}^{(n)}, \overline{u}_3^{(n)} = \overline{w}^{(n)} + y\overline{\theta}^{(n)},\end{gathered} \tag{29}$$

where $\overline{u}^{(n)}$, $\overline{v}^{(n)}$, $\overline{w}^{(n)}$ – displacement components under n–load; , $\alpha_1^{(n)}$ $\alpha_2^{(n)}$ – section angles under pure bending; , $\beta_1^{(n)}$ $\beta_2^{(n)}$ – shear angles; $\theta^{(n)}$ – torsion angle; $\nu_1^{(n)}$ – shear twist under n – load; φ – Saint –Wenans torsion function.

The strain components, according to (2), are determined by the Cauchy formula:

$$\overline{e}_{11}^{(n)} = \frac{\partial \overline{u}^{(n)}}{\partial x} - y\frac{\partial \overline{\alpha}_1^{(n)}}{\partial x} - z\frac{\partial \overline{\alpha}_2^{(n)}}{\partial x} + \varphi(y,z)\frac{\partial \overline{v}^{(n)}}{\partial x} + a_1(y,z)\frac{\partial \overline{\beta}_1^{(n)}}{\partial x} + a_2(y,z)\frac{\partial \overline{\beta}_2^{(n)}}{\partial x};$$

$$\overline{e}_{13}^{(n)} = \frac{\partial \overline{w}^{(n)}}{\partial x} + y\frac{\partial \overline{\theta}^{(n)}}{\partial x} - \overline{\alpha}_2^{(n)} + \frac{\partial \varphi}{\partial z}\overline{v}^{(n)} + \frac{\partial a_1}{\partial z}\overline{\beta}_1^{(n)} + \frac{\partial a_2}{\partial z}\overline{\beta}_2^{(n)}, \tag{30}$$

$$\overline{e}_{12}^{(n)} = \frac{\partial \overline{v}^{(n)}}{\partial x} - z\frac{\partial \overline{\theta}^{(n)}}{\partial x} - \overline{\alpha}_1^{(n)} + \frac{\partial \varphi}{\partial y}\overline{v}^{(n)} + \frac{\partial a_1}{\partial y}\overline{\beta}_1^{(n)} + \frac{\partial a_2}{\partial y}\overline{\beta}_2^{(n)}.$$

The physical equations of state for stresses and strains marked with a line are assumed to be of the type (3):
(a) When using the Moskvitin model, taking into account the damageability of the material

$$\overline{\sigma}_{11}^{(n)} = 3G(1-\omega^{(n)})\overline{e}_{11}^{(n)},\ \overline{\sigma}_{13}^{(n)} = G(1-\omega^{(n)})\overline{e}_{13}^{(n)},\ \overline{\sigma}_{12} = G(1-\omega^{(n)})\overline{e}_{12}^{(n)}; \tag{31}$$

In the case of linear strengthening, the plasticity function is represented as

$$\omega^{(n)} = \begin{cases} 0, & \text{at } \overline{\varepsilon}_u^{(n)} \le \overline{\varepsilon}_s^{(n)}(\eta) \\ \lambda_n\left[1 - \dfrac{\overline{\varepsilon}_s^{(n)}(\eta)}{\overline{\varepsilon}_u^{(n)}}\right], & \text{at } \overline{\varepsilon}_u^{(n)} > \overline{\varepsilon}_s^{(n)}(\eta) \end{cases}.$$

In the case of the generalised Mazing principle $\lambda_n = \lambda, \overline{\varepsilon}_u^{(n)} = \alpha_n \varepsilon_s$, using Gusenkov –Schneiderovich deformation diagrams $\overline{\varepsilon}_s^{(n)} = 2\varepsilon_s$, $\lambda_n = 1 - g_n$ where g_n **is** determined experimentally. When damage accumulation is taken into account [2]:

$$\overline{\varepsilon}_s^{(n)}(\eta) = \alpha_1^{n-z}(1+\alpha_1)\varepsilon_s + (3G)^{-1}B^{1/\alpha}\cdot$$
$$\cdot\left[1 - 0{,}5(1+\alpha_1)\alpha_1^{n-2}\right]\left[1-(1-\eta)^{1+\alpha}\right]^{1/\alpha}(n-1)^{-1/\alpha},$$

and the damage function η is determined from the kinetic equation

$$\frac{d\eta}{d\lambda} = f(\overline{\sigma}_u, \eta_n);\ \ f = A\cdot\frac{\left(\overline{\sigma}_u^{(n)}\right)^{\alpha}}{\left(1-\gamma\eta_n^r\right)^{\beta}}. \tag{32}$$

provided $\eta(0)=0$, $\eta(\lambda_N)=1$, where N is the number of half–cycles before reaching the limit state (failure).

b) using the Buriev model in current coordinates, taking into account variable loading:

$$\sigma_{11}^{(k)} = 3G\left\{e_{11}^{(k)} - \left[\omega^{(k)} e_{11}^{(k)} + \sum_{m=1}^{k-1} \omega^{0(k-m)} \bar{\varepsilon}_{11}^{0(k-m)}\right]\right\},$$

$$\sigma_{13}^{(k)} = G\left\{e_{31}^{(k)} - \omega^{(k)} \bar{\varepsilon}_{31}^{(k)} - \sum_{m=1}^{k-1} \omega^{0(k-m)} \bar{\varepsilon}_{31}^{0(k-m)}\right\},$$

$$\sigma_{12}^{(k)} = G\left\{e_{12}^{(k)} - \omega^{(k)} \bar{\varepsilon}_{12}^{(k)} - \sum_{m=1}^{k-1} \omega^{0(k-m)} \bar{\varepsilon}_{12}^{0(k-m)}\right\}. \quad (33)$$

By deducing the equation of motion (equilibrium) for quantities with a line, we obtain a system of differential equations with appropriate boundary and initial conditions, similar in form to (21) and (22):

$$\begin{aligned} &\tilde{A}\frac{\partial^2 Y^{(n)}}{\partial t^2} + \frac{\partial}{\partial x}\left[\left(A^{yn} - A^{n\pi}\right)\frac{\partial Y^{(n)}}{\partial x} + \left(B^{yn} - B^{n\pi}\right)Y^{(n)}\right] + \\ &+\left(C^{yn} - C^{n\pi}\right)\frac{\partial Y^{(n)}}{\partial x} + \left(D^{yn} - D^{n\pi}\right)Y^{(n)} + Q^{н} = 0; \end{aligned} \quad (34)$$

$$\left\{\left(A^{yn} - A^{n\pi}\right)\frac{\partial Y}{\partial x} + \left(B^{yn} - B^{n\pi}\right)Y + Q^{гр}\right\}\delta Y\Big|_x = 0;\ \tilde{A}\frac{dY^{(n)}}{dt}E\delta Y^{(n)}\Big|_t = 0. \quad (35)$$

Here the quadratic matrices of order nine *A, B, C, D*, the vectors of external forces of order nine $Q^{н}$ and $Q^{гр}$ and the coefficients have the following form:

$$(a_{ij} = a_{ij}^{yn} - a_{ij}^{пл(n)},\ b_{ij} = b_{ij}^{yn} - b_{ij}^{пл(n)},\ c_{ij} = -b_{ij},\ d_{ij} = d_{ij}^{yn} - d_{ij}^{пл(n)}).$$

In particular, from (34) and (35) we have the following system of differential equations of equilibrium of a rod (pipeline) under space – variable loads with boundary conditions in vector form (in current values):

$$\begin{aligned} &\frac{\partial}{\partial x}\left[\left(A^{yn} - A^{пл(k)}\right)\frac{\partial Y^{(n)}}{\partial x} + \left(B^{yn} - B^{пл(n)}\right)Y^{(n)}\right] + \left(C^{yn} - C^{пл(n)}\right)\frac{\partial Y^{(n)}}{\partial x} + \left(D^{yn} - D^{пл(n)}\right)Y^{(n)} = P^{(n)} + \\ &+\frac{\partial}{\partial x}\left(A^{пл(n)}\frac{\partial Y^{0(n-1)}}{\partial x} + B^{пл}Y^{0(n-1)}\right) + C^{пл(k)}\frac{\partial Y^{0(n-1)}}{\partial x} + D^{пл(n)}Y^{0(n-1)} + \\ &+\sum_{m=1}^{n-1}\left\{\frac{\partial}{\partial x}\left[A^{пл(n-m)}\frac{\partial}{\partial x}\left(Y^{0(n-m)} - Y^{0(n-m-1)}\right) + B^{пл(n-m)}\left(Y^{0(n-m)} - Y^{0(n-m-1)}\right)\right] + \right. \\ &\left. + C^{пл(n-m)}\frac{\partial}{\partial x}\left(Y^{0(m-m)} - Y^{0(n-m-1)}\right) + D^{пл(n-m)}\left(Y^{0(m-m)} - Y^{0(n-m-1)}\right)\right]\right\}. \end{aligned} \quad (36)$$

Boundary conditions:

$$\left\{\left(A^{yn} - A^{пл(n)}\right)\frac{\partial Y^{(n)}}{\partial x} + \left(B^{yn} - B^{пл(n)}\right)Y^{(n)} - Q^{(n)} - B^{пло(n)}Y^{(n)} - A^{пло(n)}\frac{\partial Y^{0(n-1)}}{\partial x} - \right.$$

$$-\sum_{m=1}^{n-1}\Big[A^{пл(n-m)}\frac{\partial}{\partial x}\Big(Y^{0(n-m)}-Y^{0(n-m-1)}\Big)+B^{пл(n-m)}\Big(Y^{0(n-m)}-Y^{0(n-m-1)}\Big)\Big]\Big\}\delta Y^{n}\Big|_{\Gamma}. \quad (37)$$

Here $Y^{(n)}$ are the desired vectors of the ninth – order function in the current coordinates. Expressions of internal forces and moments in vector form can be represented as

$$P^{(n)}=\frac{3Gh_0I_0}{l^3}\left\{\left(\widetilde{A}^{уп}-\widetilde{A}^{пл(n)}\right)\frac{\partial Y^{(n)}}{\partial x}+\left(\widetilde{B}^{уп}-\widetilde{B}^{пл(n)}\right)Y^{(n)}\right\}, \quad (38)$$

where $P^{(n)}$ is a twelfth – order function vector

$$P^{(n)}=\left\{N_x^{(n)},M_y^{(n)},M_z^{(n)},M_\varphi^{(n)},M_{a_1}^{(n)},M_{a_2}^{(n)},Q_1^{(n)},M_x^{(n)},Q_2^{(n)},Q_{\bar{a}_1}^{(n)},Q_{\bar{a}_2}^{(n)},M_{\bar{\varphi}}^{(n)}\right\}.$$

The matrices $\widetilde{A}^{уп},\widetilde{A}^{пл(n)},\widetilde{B}^{уп},\widetilde{B}^{пл(n)}$ of the twelfth–order square matrix and the elements are described as follows:

$$\widetilde{a}_{ij}=a_{ij},\quad \widetilde{a}_{10,s}=b_{s,5};\quad \widetilde{a}_{11,s}=b_{s,6};\quad \widetilde{a}_{12,s}=b_{s,4};\quad \widetilde{b}_{ij}=b_{ij},$$
$$\widetilde{b}_{10,r}=d_{2,s};\ \widetilde{b}_{11,r}=d_{r,6};\ \widetilde{b}_{12,r}=d_{r,4};$$
$$(i,j=1,2,...9; s=7,8,9; r=2,3,4,5,6;)$$

The finite difference method [21] and modifications of the Ilyushin elastic solution method [4] are used to solve the boundary value problem. During their approximation a central difference scheme of the second order of accuracy is used.

The vector equation (36) after applying the differences of the scheme takes the form:

$$\left(A_i^{уп}-A_i^{пл(n)}\right)Y_{i+1}^{(n)}-\left(B_i^{уп}-B_i^{пл(n)}\right)Y_i^{(n)}+\left(C_i^{уп}-C_i^{пл(n)}\right)Y_i^{(n)}=\vec{Q}_i^{(n)}+\vec{Q}_i^{пл}+\vec{Q}^{пл0} \quad (39)$$

To solve the formulated algebraic equations (30) with the corresponding boundary conditions, a run–time method is used using the following recurrence formula:

$$V_i=\alpha_iV_{i+1}+\beta_i;\quad i=N-1,...,1. \quad (40)$$

Here $\alpha_i=(B_i-C_i\alpha_{i-1})^{-1}A_i;\ \beta_i=(B_i-C_i\alpha_{i-1})^{-1}(C_i\beta_{i-1}-F);\quad i=1,2,...,N-1.$

3 Results and discussion

First, let us consider the results of calculation of thin – walled rods of rectangular cross–section, buttressed at the ends under repeated – variable loading according to the generalized principle of Masing – Moskvitin [21-23]. We give results of calculation of thin – walled bars of rectangular cross – section, pinched at the ends under repeated alternating loading. The problem is solved with the following input data: geometrical and mechanical characteristics of the rod:

$$l = 250\ sm,\ h_0 = 10\ sm,\ b_0 = 10\ sm,$$
$$E = 2 \cdot 10^6 \frac{kg}{sm^2},\ \sigma_S = 3210 \frac{kg}{sm^2},\ \varepsilon_S = 0.0015;$$

uniformly distributed external loads: $f_0^+ = 25$; $f_0^- = 50$; $\bar{f}_0^+ = 10$; $\bar{f}_0^- = 5 \frac{kg}{sm^2}$;

$$\bar{\gamma} = \frac{\pi}{4}; \alpha = \frac{\pi}{3}; \gamma^* = \frac{\pi}{6}; \alpha^* = \frac{\pi}{2}; q^{(n)} = (-1)^{n+1}.$$

Table 1 shows the maximum values of displacement vector $\vec{V}_i^{(n)}$ along the length of the rod under cyclic loading (n=1, 2, 3, 4, 5) according to the generalized principle of Masing –Moskvitin for different materials (B – 96, D –16T and St TS).

The variable loading theorem was used to determine the true values of the calculated values. The condition for the occurrence of secondary, tertiary, etc. plastic regions is $\bar{\sigma}_u^{(n)} \geq \alpha_n \sigma_s$, where α_n is the scale factor.

Table 1. Maximum displacement vector values

Displacement vector ($\vec{V}_i^{(n)}$)	Number of load (n)	Cyclically hardening		Cyclically unstrengthening
		B-96 Q=2.08; æ=0.047	D-16T Q=2.02; æ=0.03	St TS Q=1.93; æ=0.011
1	**2**	**3**	**4**	**5**
$W^{(1)}(0.5)$	1	-0.27217078	-0.27217078	-0.27217078
$\bar{W}^{(n)}(0.5)$	2	0.54432625	0.54433769	0.54435589
	3	-0.54431336	-0.54432956	-0.54435274
	4	0.54430573	0.54432480	0.54435092
	5	-0.54430034	-0.54432138	-0.54434963
$W^{(5)}(0.5)$		-0.27215250	-0.27215924	-0.27216635
$\alpha_1^{(n)}(0.2)$	1	-0.82530072	-0.82530072	-0.82530072
$\bar{\alpha}_1^{(n)}(0.2)$	2	1.65055491	1.65058966	1.65064538
	3	-1.65051591	-1.65056491	-1.65063576
	4	1.65049275	1.65055051	1.65063016
	5	-1.65047651	-1.65054020	-1.65062621
$\alpha_1^{(5)}(0.2)$		-0.82524549	-0.82526567	-0.82528715
$\beta_1^{(n)}(0.1)$	1	-0.01985111	-0.01985111	-0.01985111
$\bar{\beta}_1^{(n)}(0.1)$	2	0.03970113	0.03970194	0.03970327
	3	-0.03970023	-0.03970136	-0.03970304
	4	0.03969970	0.03970103	0.03970291
	5	-0.03969933	-0.03970079	-0.03970282
$\beta_1^{(5)}(0.1)$		-0.01984984	-0.01985029	-0.01985079
$V^{(n)}(0.5)$	1	-0.25519159	-0.25519159	-0.25519159
$\bar{V}^{(n)}(0.5)$	2	0.51036890	0.51037953	0.51039654
	3	-0.51035692	-0.51037199	-0.51039365
	4	0.51034984	0.51036755	0.51039197
	5	-0.51034482	-0.51036438	-0.51039079
$V^{(5)}(0.5)$		-0.25517460	-0.25518087	-0.25518752

The variable loading theorem was used to determine the true values of the calculated values. It should be noted that the values of the calculated values obtained using the variable loading theorem and the relations linking the stress and strain components in the current coordinates using the generalised Masing principle coincide.

As a second example, the calculation of thin – walled rods of rectangular cross – section clamped at the ends under alternating loading with allowance for damage accumulation is performed. The problem is solved with the following initial data: material constants of the kinetic damage equation: $A=1.2\cdot10^{-4}$; $\alpha=\beta=5$; $\gamma=0.8$; α_1=0.97; $B=1.4\cdot10^3$; ε_s=0.0015. Calculation results are given (y=0; z=b_0) at cross – sectional points of the rod; $x=0.0$; $x=0.2$; $x=0.5$ under cyclic loading. Table 2 shows the kinetics of change in ductility function $\omega^{(n)}$ and damageability $\eta^{(n)}$, as well as strain intensity $\bar{\varepsilon}_u^{(n)}(\eta)$ and stresses $\bar{\sigma}_u^{(n)}(\eta)$ as a function of cyclic loading.

Table 2. Influence of damageability on plasticity parameters

n	x	$\omega^{(n)}$	$\omega^{(n)}(\eta)$	$10^2\eta^{(n)}$	$10^2\bar{\varepsilon}_u^{(n)}(\eta)$	$10^{-2}\bar{\sigma}_u^{(n)}(\eta)$
$n=1$	0.0	0.8677	0.8677	0.0000	1.7333	4.7733
	0.2	0.7422	0.7422	0.0000	0.6859	3.7259
	0.5	0.7160	0.7160	0.0000	0.6091	3.6491
$n=2$	0.0	0.7914	0.7881	0.0445	1.7340	7.7228
	0.2	0.5492	0.5409	0.0256	0.6862	6.6750
	0.5	0.4987	0.4893	0.0231	0.6094	6.5982
$n=5$	0.0	0.7889	0.8021	1.8118	1.7334	7.1994
	0.2	0.5429	0.5765	1.2964	0.6860	6.1520
	0.5	0.4916	0.5294	1.2193	0.6092	6.0752
$n=6$	0.0	0.7885	0.8066	2.2884	1.7341	7.0361
	0.2	0.5420	0.5878	1.6284	0.6862	5.9883
	0.5	0.4906	0.5422	1.5302	0.6094	5.9115
$n=9$	0.0	0.7876	0.8191	3.4466	1.7335	6.5727
	0.2	0.5398	0.6193	2.4125	0.6860	5.5252
	0.5	0.4881	0.5776	2.2606	0.6092	5.4484
$n=10$	0.0	0.7875	0.8231	3.7556	1.7341	6.4282
	0.2	0.5394	0.6293	2.6155	0.6862	5.3803
	0.5	0.4876	0.5889	2.4487	0.6094	5.3035

Figure 2 shows the calculated values of $W^{(n)}$ and $\alpha_1^{(n)}$ obtained considering the secondary plastic deformations and elastic unloading. Note that the residual calculated values differ significantly from the values calculated from the elastic unloading theorem.

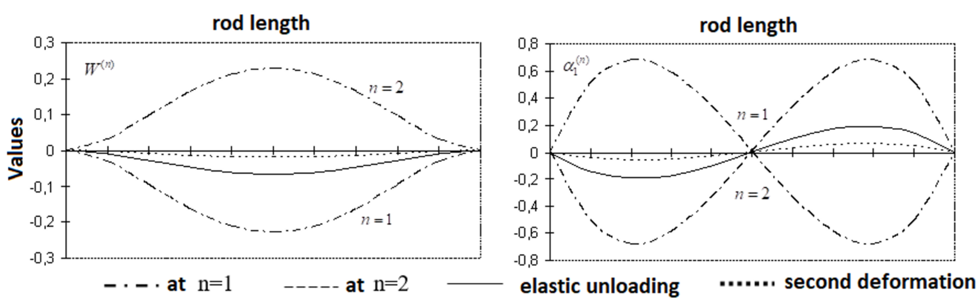

Fig. 2. Determination of residual displacement vector values $W^{(k)}$ and $\alpha_1^{(k)}$

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For different load intensities ($\delta = 1, 1.5, 2$) the variations of displacements $W^{(n)}$, $\alpha_1^{(n)}$ and moments $M_y^{(n)}$, $M_z^{(n)}$ along the length of the rod are shown in Fig. 3 and 4.

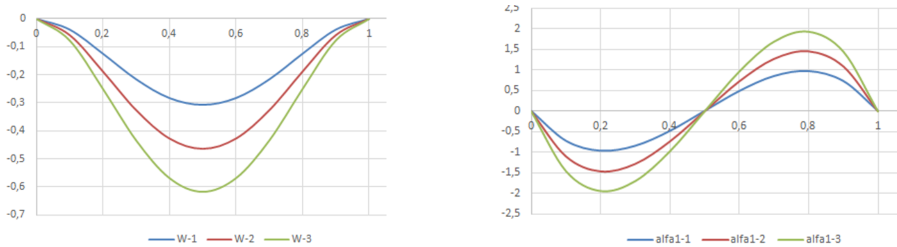

Fig. 3. Displacement variations along the length of the rod

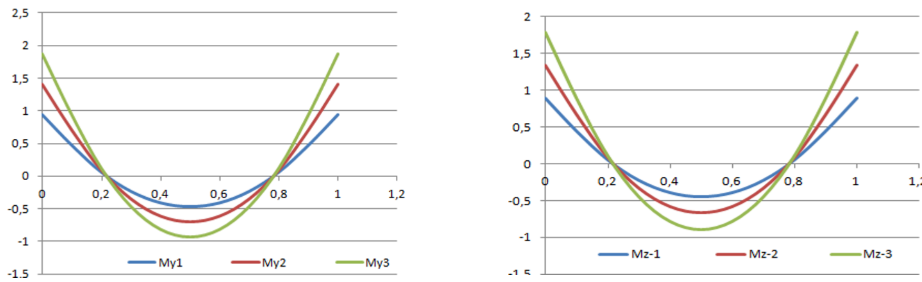

Fig. 4. Changes of moments along the length of the rod

The analysis of numerical experiment shows that the values of plasticity function and damage zone change with increasing number of loading cycles. This, in turn, affects the kinetics of displacements, forces and moments under alternating loading of elastoplastic structural elements.

4 Conclusions

A system of differential equations of motion (equilibrium) for thin–walled rods and pipelines is formulated and on the basis of variational principle the boundary problems under spatially variable loading are generated. The kinetics of stress – strain state of structural elements under alternating loading with consideration of generalized Mazing's principle and damageability of material is investigated. Influence of cyclic diagrams of deformation, secondary plastic deformations and elastic unloading on calculated values is also shown.

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