On the unique solvability of a nonlocal boundary value problem with the poincaré condition

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Abstract. As is known, it is customary in the literature to divide degenerate equations of mixed type into equations of the first and second kind. In the case of an equation of the second kind, in contrast to the first, the degeneracy line is simultaneously the envelope of a family of characteristics, i.e. is itself a characteristic, which causes additional difficulties in the study of boundary value problems for equations of the second kind. In this paper, in order to establish the unique solvability of one nonlocal problem with the Poincaré condition for an elliptic-hyperbolic equation of the second kind developed a new principle extremum, which helps to prove the uniqueness of resolutions as signed problem. The existence of a solution is realized by reducing the problem posed to a singular integral equation of normal type, which known by the Carleman-Vekua regularization method developed by S.G. Mikhlin and M.M. Smirnov equivalently reduces to the Fredholm integral equation of the second kind, and the solvability of the latter follows from the uniqueness of the solution delivered problem.

1 Introduction

Boundary value problems for degenerate equations of elliptic and equations of mixed types are in the center of attention of mathematicians and mechanics due to the presence of numerous applications in the study of problems in mechanics, physics, engineering and biology. Starting from [1], [2], a new direction has appeared in the theory of equations of elliptic and mixed types, in which nonlocal boundary value problems (problems with a shift) and Bitsadze-Samarskii problems are considered. Further, it turned out that non-local boundary conditions arise in problems of predicting soil moisture [3], in modeling fluid filtration in porous media [4], in mathematical modeling of laser radiation processes and problems of plasma physics [5], as well as in mathematical biology [6].

Solving various boundary value problems with the Poincaré conditions or with a conormal derivative for the Tricomi, Lavrentiev-Bitsadze and more general equationsdevoted to a large number of articles [713]. We note that the results of all the

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listed papers were obtained for equations of the first kind, and for equations of the second kind, nonlocal boundary value problems with the Poincaré condition have not been previously studied. Therefore, the study of non-local boundary value problems with a conormal derivative for equations of mixed elliptic-hyperbolic type of the second kindseems to be very relevant and little studied. Note the works [14,15]. In this paper, we study a nonlocal boundary value problem with the Poincaré condition for an elliptic-hyperbolic type equation of the second kind, i.e. for an equation where the line of degeneracy is a characteristic.

2 Statement of the problem

Consider the equation

$$sgny|y|^m u_{xx} + u_{yy} = 0, \qquad m \in (0;1) \tag{1}$$

Let Ω is a finite simply connected region of the plane of independent variables *x, y*, bounded at $y > 0$ crooked σ dot ends $A(0,0)$, $B(1,0)$ and segment $AB(y=0)$, and when $y < 0$ characteristics

$$AC:\ x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = 0, \quad BC:\ x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = 1$$

equations (1).

Let further $\Omega_1 = \Omega \cap \{\, y > 0\}$, $\Omega_2 = \Omega \cap \{\, y < 0\}$,

$J = \{(x,y): 0 < x < 1,\ y = 0\}$, $\Omega = \Omega_1 \cup \Omega_2 \cup J$, $2\beta = m/(m+2)$, and

$$\beta \in (-0{,}5;0). \tag{2}$$

Problem. C. Required find function $u(x,y)$, which has the following properties:

1) $u(x,y) \in C(\overline{\Omega}) \cup C^1(\Omega_1 \cup \Omega_2 \cup \sigma \cup J)$, and the derivatives u_x and u_y can address infinity of order less than one at points $A(0,0)$ and $B(1,0)$;

2) $u(x,y) \in C^2(\Omega_1)$ is a regular solution of equation (1) in the domain Ω_1, and in the region Ω_2 is a generalized solution from the class R_2 [16];

3) the gluing condition is satisfied on the degeneracy line

$$\lim_{y \to +0} u_y(x,y) = -\lim_{y \to -0} u_y(x,y) \tag{3}$$

4) satisfies the following boundary conditions

$$\left\{\delta(s) A_s[u] + \rho(s) u\right\}\Big|_{\sigma} = \varphi(s), \qquad 0 < s < l, \tag{4}$$

$$\frac{d}{dx} u\left[\Theta_0(x)\right] + b\frac{d}{dx} u\left[\Theta_1(x)\right] = c(x), \qquad (x,0) \in J, \tag{5}$$

where l – the length of the whole curve σ, s – are length σ, counted from the point $B(1,0)$, a $\rho(s)$, $\delta(s)$, $\varphi(s)$, $c(x)$ – given functions, and $b = const \neq 0$,

$$\rho(s)\delta(s) \geq 0, \quad 0 \leq s \leq l, \tag{6}$$

$$\rho(s), \delta(s), \varphi(s) \in C[0,l], \qquad c(x) \in C^1[0,1] \cap C^2(0,1), \tag{7}$$

here

$$\Theta_0\left(\frac{x}{2}, -\left(\frac{m+2}{4}x\right)^{2/(m+2)}\right) \text{and} \ \Theta_1\left(\frac{x+1}{2}; -\left(\frac{m+2}{4}(1-x)\right)^{\frac{2}{m+2}}\right) \tag{8}$$

- points of intersection of the characteristics of equation (1), emerging from the points $x \in J$, with characteristics AC and BC respectively, and $A_s[u]$ determined from the formula

$$A_s[u] \equiv y^m \frac{dy}{as}\frac{\partial u}{\partial x} - \frac{dx}{as}\frac{\partial u}{\partial y}.$$

Note that if $\delta(s) \equiv 0, \quad b = 0$, then the tasks C matches the tasks T studied in [17]. Therefore, in what follows, we will assume that $\delta(s) \neq 0$.

Uniqueness of solutions to the problem C.

To prove the uniqueness of the solution to the problem C. The following lemmas play an important role.

Lemma 1. If the function $\tau'(x)$ satisfies Hölder's condition with exponent $k > -2\beta$ at $0 < x < 1$, then the function

$$T(x) = \frac{1}{\Gamma(1-2\beta)} D_{0x}^{1-2\beta}\tau(x) \tag{9}$$

can be represented as

$$T(x) = \frac{sin2\pi\beta}{2\pi\beta}\frac{d}{dx}\int_0^x (x-t)^{2\beta}\tau'(t)dt.$$

Lemma 2. Let the conditions

$$\tau(x) \in C[0,1] \cap C^{(1,k)}(0,1), k > -2\beta \tag{10}$$

and function $\tau(x)$ at the point $x = x_0$ $\left(x_0 \in (0,1)\right)$ takes on the largest positive value (LPV) and the smallest negative value (SNV). Then the function

$$E(x)=\int_0^1 \frac{(1-t)^{-2\beta}T(t)}{x-t}dt$$

at the point $x=x_0$ can be represented as

$$E(x_0)=(1-x_0)^{-2\beta}\left\{\left[x_0^{\ 2\beta-1}cos2\beta\pi+(1-x_0)^{2\beta-1}\right]\tau(x_0)-\tau(1)(1-x)^{2\beta-1}+\right.$$
$$\left.+(1-2\beta)\left[cos2\beta\pi\int_0^{x_0}\frac{\tau(x_0)-\tau(t)}{(x_0-t)^{2-2\beta}}d\,t-\int_{x_0}^1\frac{\tau(t)-\tau(x_0)}{(t-x_0)^{2-2\beta}}d\,t\right]\right\} \qquad (11)$$

Lemma 3. Let conditions (2), (10) be satisfied and the function $\tau(x)$ at the point $x=x_0$ $\left(x_0\in(0,1)\right)$ accepts refineries (SNV). Then the function $T(x)$ (see (9)) at the point $x=x_0$ can be represented as

$$T(x_0)\equiv\frac{1}{\Gamma(1-2\beta)}D_{0x}^{1-2\beta}\tau(x)\Big|_{x=x_0}=$$
$$=\frac{Sin2\beta\pi}{\pi}\left[x_0^{\ 2\beta-1}\tau(x_0)+(1-2\beta)\int_0^{x_0}\frac{\tau(x_0)-\tau(t)}{(x_0-t)^{2-2\beta}}d\,t\right]$$

and

$$T(x_0)<0\ \ \left(T(x_0)>0\right),\quad x_0\in J \qquad (12)$$

Proof of Lemma 1-3is carried out in the same way as in [22].

Lemma 1-3 implies the following.

Theorem 1. (An analogue of the extremum principle of A.V. Bitsadze). If conditions (2) are satisfied and $b<0$, then the solution $u(x,y)$ problem C at $c(x)\equiv 0$ and $\tau(1)=0$ own refinery and SNV in a closed area $\overline{\Omega}_1$ only reaches $\bar{\sigma}$.

Proof of Theorem 1. Indeed, due to the extremum principle for elliptic equations [5], [23], the solution $u(x,y)$ equations (1) inside the region Ω_1 cannot reach its refinery and SNV. Let us show that the solution $u(x,y)$ equation (1) does not reach its OR and SNV on the segment J. Assume the opposite, let $u(x,y)$ some point $(x_0,0)$ segment J reaches its refinery (SNV). Based on Lemma 2, if the function $\tau(x)$ at the point $(x_0,0)$ accepts the refinery (SNV), then $A(x)$ at the point $x=x_0$ can be represented in the form (11), and

$$E(x_0)>0\ \ \left(E(x_0)<0\right),\quad (x_0,0)\in J \qquad (13)$$

Now let's define the sign $\nu^-(x)$ at the point $(x_0,0)\in J$. Due to (12) and (13) at $c(x)\equiv 0$ we get

$$\nu^{-}(x_0)<0 \;\; (\nu^{-}(x_0)>0), \quad (x_0,0)\in J \tag{14}$$

But on the other hand, by virtue of the Zaremba-Giraud principle [24], [26], for the solution of equation (1), taking into account (15), we have

$$\nu^{+}(x_0)<0 \;\; (\nu^{+}(x_0)>0), \quad (x_0,0)\in J \tag{15}$$

Taking into account (4) from (14) we find

$$\nu^{+}(x_0)>0 \;\; (\nu^{+}(x_0)<0), \qquad (x_0,0)\in J$$

This inequality contradicts inequality (15). In this way, $u(x,y)$ does not reach its refinery (SNV) in the open section J. Theorem 1 is proved.

Theorem 2. If the conditions of Theorem 1 are satisfied, and the functions $\delta(s)$ and $\rho(s)$ near points $A(0,0)$, $B(1,0)$ satisfy conditions (7) and

$$\rho(0)\neq 0, \;\; \rho(l)\neq 0 \tag{16}$$

$$\left|\delta(s)\right| \le const\left[s\;(l-s)\right]^{\varepsilon_0-\frac{m^2+2m-2}{m+2}}, \quad -1<m<0, \;\; \varepsilon_0=const>0, \tag{17}$$

then in the area D there cannot be more than one solution to the problem C.

Proof of Theorem 2. Let $\varphi(s)\equiv c(x)\equiv 0$, then, by virtue of Theorem 1, it suffices to show that the solution of the problem C cannot reach its positive maximum and negative minimum on σ.

Assume that a positive maximum (negative minimum) is reached at some point $s_0\in\sigma$, different from the points $A(0,0)$ and $B(1,0)$. Then at this point, due to the Zaremba-Giraud principle [24, 27] $A_{s_0}\;[u]>0\left(A_{s_0}\;[u]<0\right)$, and the boundary condition (5) takes the form

$$A_{s_0}[u]=-\frac{\rho(s_0)\delta(s_0)}{\delta^2(s_0)}u$$

But this is impossible due to condition (7).

Therefore, at interior points σ function $u(x,y)$ does not reach its positive maximum (negative minimum).

At points $A(0,0)$ and $B(1,0)$, taking into account (2), (3), (17) we have respectively.

$$\lim_{S\to 0}\delta(s)A_s[u]=0 \text{ and } \lim_{S\to l}\delta(s)A_s[u]=0 \tag{18}$$

If a positive maximum (negative minimum) is reached at the point $A(0,0)$ or $B(1,0)$, then by virtue of (18) the boundary condition (5) takes the form

$$\rho(0)\, u(0,0)=0 \text{ or } \rho(l)\, u(1,0)=0$$

Hence, taking into account (16), we obtain

$$u(A)=u(0,0)=\tau(0)=0, \quad u(B)=u(1,0)=\tau(1)=0. \tag{19}$$

Means, $u(x,y)$ does not reach a positive maximum (negative minimum) at points $A(0,0)$ and $B(1,0)$. In this way, $u(x,y)$ does not reach a positive maximum (negative minimum) on the curve $\overline{\sigma}$.

Based on the extremum principle (see Theorem 1), we conclude that $u(x,y)=const$ in $\overline{\Omega}_1$. Therefore, taking into account (19), we have $u(x,y)\equiv 0$ in $\overline{\Omega}_1$. Due to the uniqueness of the solution of the Cauchy problem in the domains Ω_{2j} $(j=\overline{1,3})$ for equation (1), we obtain that $u(x,y)\equiv 0$ in $\overline{\Omega}_{2j}$ $(j=\overline{1,3})$. Hence it follows that $u(x,y)\equiv 0$ in $\overline{\Omega}$. This proves the uniqueness of the solution of the problem C. Theorem 2.is proved.

Existence of a solution to the problem C

When studying the problem C an important role is played by the functional relationships between $\nu^{\pm}(x)$ and $\tau(x)$ from the elliptic and hyperbolic parts of the domain Ω, where

$$u(x,0)=\tau(x), \quad (x,0)\in \bar{J}, \tag{20}$$

$$\lim_{y\to -0}\frac{\partial u(x,y)}{\partial y}=\nu^{-}(x), \qquad \lim_{y\to +0}\frac{\partial u(x,y)}{\partial y}=\nu^{+}(x), \quad (x,0)\in J. \tag{21}$$

Generalized solution of the Cauchy problem with data (20), (21) for equation (1) from the class R_2 in the area of Ω_2 is given by the formula [16], [3]:

$$u(\xi,\eta)=\int_0^{\xi}(\eta-t)^{-\beta}(\xi-t)^{-\beta}T(t)\,dt+\int_{\xi}^{\eta}(\eta-t)^{-\beta}(t-\xi)^{-\beta}N(t)\,dt, \tag{22}$$

Where

$$\xi=x-\frac{2}{m+2}(-y)^{\frac{m+2}{2}},\ \eta=x+\frac{2}{m+2}(-y)^{\frac{m+2}{2}},\ \gamma_2=\left[2(1-2\beta)\right]^{2\beta-1}\frac{\Gamma(2-2\beta)}{\Gamma^2(1-\beta)},$$

$$N(t)=T(t)/2\cos\pi\beta-\gamma_2\nu^{-}(t), \tag{23}$$

$$\tau(x)=\int_0^{x}(x-t)^{-2\beta}T(t)\,dt, \tag{24}$$

Functions $T(x)$ and $\nu^{-}(x)$ continuous in $(0,1)$ and integrable on $[0,1]$, a $\tau(x)$ vanishes on the order of at least -2β at $x\to 0$.

Putting $\xi=0, \eta=x$ and $\xi=x, \eta=1$ respectively, in (22), taking into account (8), after some transformations we obtain

$$u\left[\Theta_0(x)\right]=\int_0^x (x-t)^{-\beta}t^{-\beta}N(t)dt, \tag{25}$$

$$u\left[\Theta_1(x)\right]=\int_0^x (x-t)^{-\beta}(1-t)^{-\beta}T(t)dt+\int_x^1 (t-x)^{-\beta}(1-t)^{-\beta}N(t)dt. \tag{26}$$

We put (25) and (26) in the boundary condition (6), by virtue of the fractional integration operators and (23) we obtain a functional relation between $T(x)$ and $\nu^-(x)$, transferred from the area Ω_2 on the J:

$$\gamma_1\left(x^{-2\beta}-2b\cos\pi\beta\cdot x^{-\beta}(1-x)^{-\beta}+b^2(1-x)^{-2\beta}\right)\nu^-(x)-\frac{x^{-2\beta}+b^2\cos 2\pi\beta(1-x)^{-2\beta}}{2\cos\pi\beta}T(x)-$$
$$-\frac{b^2\sin\pi\beta}{\pi}\int_0^1\frac{(1-t)^{-2\beta}T(t)}{x-t}dt=-\frac{x^{-\beta}}{\Gamma(1-\beta)}D_{0x}^{-\beta}c(x)+\frac{b(1-x)^{-\beta}}{\Gamma(1-\beta)}D_{x1}^{-\beta}c(x). \tag{27}$$

The solution of the problem DK with conditions (5) and (20) for equation (1) in the region D_1 exists, is unique and can be represented in the form [16. see (10.78)]:

$$u(x,y)=\int_0^1\tau(\xi)\frac{\partial}{\partial\eta}G_2(\xi,0;x,y)d\xi+\int_0^l\frac{\varphi(s)}{\delta(s)}G_2(\xi,\eta;x,y)ds \tag{28}$$

Where $G_2(\xi,\eta;x,y)-$ Green's function of problem DK for equation (1) [16]:
Differentiating with respect to y equation (28), then directing y to zero we get the functional relation between $\tau(x)$ and $\nu^+(x)$, transferred from the area Ω_1 on the J:

$$\nu^+(x)=\frac{k_2}{1-2\beta}\frac{d}{dx}\left[-\int_0^x(x-t)^{2\beta-1}\tau(t)dt+\int_x^1(t-x)^{2\beta-1}\tau(t)dt\right]-k_2\int_0^1\frac{\tau(t)dt}{(t+x-2tx)^{2-2\beta}}+$$
$$+\int_0^1\tau(t)\frac{\partial^2 H_2(t,0;x,0)}{\partial\eta\partial y}dt+\int_0^l\chi(s)\frac{\partial q_2(\xi(s),\eta(s);x,0)}{\partial y}ds, \tag{29}$$

where $\chi(s)$ is a solution to the integral equation

$$\chi(s)+2\int_0^l\chi(t)\Big\{A_s\left[q_2\left(\xi(t),\eta(t);x(s),y(s)\right)\right]+$$
$$+\frac{\rho(s)}{\delta(s)}q_2\left(\xi(t),\eta(t);x(s),y(s)\right)\Big\}dt=\frac{2\varphi(s)}{\delta(s)}.$$

And $q_2(\xi,\eta,x,y)$ is the fundamental solution of equation (1) and it has the form:

$$q_2(\xi,\eta,x,y)=k_2\left(\frac{4}{m+2}\right)^{4\beta-2}\left(r_1^2\right)^{-\beta}(1-w)^{1-2\beta}F(1-\beta,1-\beta,2-2\beta;1-w)$$

where

$$\left.\begin{matrix} r^2 \\ r_1^2 \end{matrix}\right\}=(\xi-x)^2+\frac{4}{(m+2)^2}\left(\eta^{\frac{m+2}{2}}\mp y^{\frac{m+2}{2}}\right)^2,$$

$$w=\frac{r^2}{r_1^2},\quad \beta=\frac{m}{2(m+2)},\quad -\frac{1}{2}<\beta<0,\ k_2=\frac{1}{4\pi}\left(\frac{4}{m+2}\right)^{2-2\beta}\frac{\Gamma^2(1-\beta)}{\Gamma(2-2\beta)},$$

$F(a,b,c;z)$ is hypergeometric function of Gauss [23].

Substituting (24) into (29) and taking into account some identities of fractional differential operators, we obtain a functional relation between $T(x)$ and $\nu^+(x)$, transferred from the area Ω_1 on the J:

$$\nu^+(x)=-\frac{\pi k_2 tg\beta\pi}{1-2\beta}T(x)-\frac{k_2}{1-2\beta}\int_0^1\left(\frac{1-t}{1-x}\right)^{-2\beta}\left[\frac{1}{t-x}+\frac{1-2t}{x+t-2xt}\right]T(t)dt+$$
$$+\int_0^1 T(t)dt\int_t^1(z-t)^{-2\beta}\frac{\partial^2 H_2(z,0;x,0)}{\partial\eta\partial y}dz-\frac{2k_2}{1-2\beta}\int_0^1\left(\frac{1-t}{1-x}\right)^{-2\beta}\frac{T(t)dt}{1-2x}+$$
$$+\int_0^l\frac{\partial q_2(\xi(s),\eta(s);x,0)}{\partial y}\chi(s)ds,\quad (x,0)\in J\cdot \qquad (30)$$

Theorem 3. If conditions (2), (3), and (7) are satisfied, then in the region Ω the solution of the problem C exists.

Proof of Theorem 3. Excluding $\nu^{\pm}(x)$ from relations (27) and (30), taking into account (4) and (24), we obtain a singular integral equation of the form:

$$P_1(x)T(x)+\frac{P_2(x)}{\pi i}\int_0^1\left(\frac{1-t}{1-x}\right)^{-2\beta}\left[\frac{1}{t-x}+\frac{1-2t}{x+t-2xt}\right]T(t)dt-\int_0^1K(x,t)T(t)dt=$$
$$=F(x),\quad 0<x<1,\qquad (31)$$

Where

$$P_1(x)=\frac{\pi k_2 tg\beta\pi}{1-2\beta}d_1(x)-\frac{1}{2\cos\pi\beta}d_2(x),\ P_2(x)=\frac{\pi i k_2}{1-2\beta}d_1(x)-ib^2\sin\pi\beta(1-x)^{-2\beta},$$

$$K(x,t)=d_1(x)\int_t^1(z-t)^{-2\beta}\frac{\partial^2 H_2(z,0;x,0)}{\partial\eta\partial y}dz-\frac{b^2\sin\pi\beta}{\pi}\frac{(1-2t)(1-t)^{-2\beta}}{x+t-2xt}-\frac{2k_2d_1(x)}{(1-2\beta)(1-2x)}\left(\frac{1-t}{1-x}\right)^{-2\beta}$$

$$F(x)=d_1(x)\int_0^l\frac{\partial q_2(t,\eta;x,0)}{\partial y}\chi(s)ds-\frac{x^{-\beta}}{\Gamma(1-\beta)}D_{0x}^{-\beta}c(x)+\frac{b(1-x)^{-\beta}}{\Gamma(1-\beta)}D_{x1}^{-\beta}c(x),$$

equation (31) is an equation of normal type [23, 24].

Applying the well-known Carleman-Vekua regularization method [23], we obtain the Fredholm integral equation of the second kind, the solvability of which follows from the uniqueness of the solution of the problem C. Theorem 3 is proved.

3 Conclusions

Thus, with the help of the new extremum principle developed by the authors of the article for an equation of the second kind, the uniqueness of the problem posed is proved. When studying the existence of a solution to the problem under study with the help of functional relations, a singular integral equation of normal type is obtained, the solvability of which follows from the uniqueness of the solution to the problem. The article presents new mathematical results. Which are of interest to a person skilled in the art. What can be used to build some models of gas and hydrodynamic processes, when predicting soil moisture, when modeling fluid filtration in porous media.

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