Dirichlet's problem for a third-order parabolic-hyperbolic type equation of the second kind

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Abstract. Equations of mixed type, the degeneracy line of which is the envelope of a family of characteristics, therefore, is itself also a characteristic, in the literature it is customary to call equations of mixed type of the second kind, which causes additional difficulties in the study of boundary value problems for equations of the second kind. In the present work, a boundary value problem for a homogeneous equation of parabolic-hyperbolic type of the third order of the second kind is investigated. Necessary and sufficient conditions for the existence and uniqueness of a generalized solution of the problem are found. In some special cases, the representation of the solution of the problem under study is written out explicitly.

1 Introduction

At present, boundary value problems for equations of mixed type have become an important part of the modern theory of partial differential equations. One of the main problems in the theory of partial differential equations is the study of mixed type equations, which is of both theoretical and practical interest. In 1959, I.N. Vekua pointed out the importance of the problem of equations of mixed type in connection with problems in the theory of infinitesimal bendings of surfaces. The problem of the outflow of a supersonic jet from a vessel with flat walls is reduced to the Tricomi problem for the Chaplygin equation (a degenerate equation of mixed type). There are a number of works by F. Tricomi, S. Gelderstedt, A. V. Bitsadze, M. S. Salakhitdinov, T.D. Dzhuraev and their students in which the main mixed boundary value problems are studied and new correct problems are posed for the equations of the elliptic-hyperbolic, parabolic-hyperbolic types of the first kind, i.e. equations for which the degeneracy line is not a characteristic.
In recent years, a large number of papers have appeared devoted to the study of equations of composite and mixed-composite types. Correct boundary value problems for equations of mixed-composite type, the main part of which contains an elliptic-hyperbolic operator, were first formulated by A.V. Bitsadze (see [1, 2]). These problems and some of their generalizations have now been studied in some detail. We note that the results of all the

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above works were obtained for equations of the first kind, and for equations of the second kind of the third order, boundary value problems have not been previously studied.
Therefore, the study of boundary value problems for mixed type equations of the second kind seems to be very relevant and little studied. We note the works [3-6]. In this paper, we study a local boundary value problem for equations of mixed composite type of the second kind, i.e. for an equation where the line of degeneracy is a characteristic.

2 Statement of the problem

Consider the equation

$$\frac{\partial}{\partial y}(Lu)=0, \tag{1}$$

in the domain of $D=D_1\cup D_2\cup OB$, and the domain D_2 limited at $x<0$ characteristics

$$OC: y-\frac{2}{m+2}(-x)^{\frac{m+2}{2}}=0,\ BC: y+\frac{2}{m+2}(-x)^{\frac{m+2}{2}}=1,\ OB: x=0$$

Equations

$$Lu\equiv\frac{1+\operatorname{sgn}x}{2}\left(\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial y}\right)+\frac{1-\operatorname{sgn}x}{2}\left(\frac{\partial^2 u}{\partial x^2}-(-x)^m\frac{\partial^2 u}{\partial y^2}\right)=0 \tag{2}$$
$$m\in\left(-\frac{16}{9};-\frac{7}{4}\right);$$

and the domain D_1 at $x>0$ limited by segments OA, AD, BD, OB of straight lines $y=0,\ x=1,\ y=1,\ x=0$ respectively.
The general solution of equation (1) can be represented as [5]:

$$u(x,y)=z(x,y)+\omega(x) \tag{3}$$

where $z(x,y)$ is regular solution of equation (2) in the domain D_1, and in the domain D_2 is a generalized solution of the class R. Denote $\omega(x)$ in the following form:

$$\omega(x)=\begin{cases}\omega_1(x) & at \quad x>0\\ \omega_2(x) & at \quad x<0\end{cases}$$

and $\omega_1(x)$ has all the derivatives in equation (1) and the smoothness of the function is given by the definition of a generalized solution of the class R of equation (1).
Dirichlet problem. Required to define a function $u(x,y)$, which has the following properties:
a) $u(x,y)\in C(\overline{D})$;

b) function $u(x, y)$ is a generalized solution of equation (1) of class R in the domain D_2, and in the domain D_1 is regular;
c) the gluing condition is satisfied on the degeneracy line

$$-\lim_{x\to -0}\frac{\partial u}{\partial x}=\lim_{x\to +0}\frac{\partial u}{\partial x};$$

d) u_x continuous up to the transition line both on the left and on the right;
e) satisfies the boundary conditions

$$\begin{aligned} &u|_{OA}=\tau_1(x),\ u|_{AD}=\psi(y),\ u|_{BD}=\psi_1(x),\\ &[u-w(x)]|_{OC}=\psi_2(x),\ u|_{BC}=\psi_3(x), \end{aligned}$$

Where $\tau_1(x), \psi(x), \psi_1(y), \psi_2(x), \psi_3(x)$ are given sufficiently smooth functions, and $\tau_1(0)=\psi_2(0),\ \tau_1(1)=\psi(0),\ \psi(1)=\psi_1(1),\ \psi_1(0)=\psi_3(0)$.

$$\psi_2\left(-\left(\frac{m+2}{4}\right)^{\frac{2}{m+2}}\right)=\psi_3\left(-\left(\frac{m+2}{4}\right)^{\frac{2}{m+2}}\right)$$

here $-\left(\frac{m+2}{4}\right)^{\frac{2}{m+2}}$ is coordinate of point C to x.

Note that this problem in the case $m=0$ studied in [2], and in the case $-1<m<0$ considered in [1].

Without loss of generality, we can assume that $w(0)=0,\ w(1)=1$. Based on (3) and boundary conditions, the Dirichlet problem is reduced to the definition of a regular solution in the domain D_1, a generalized solution of the class R in the domain D_2 equation (2) satisfying the conditions

$$\begin{aligned} &z|_{OA}=\tau_1(x)-w_1(x),\ \ z|_{AD}=\psi(y),\ \ z|_{BD}=\psi_1(x)-w_1(x),\\ &z|_{OC}=\psi_2(x),\qquad z|_{BC}=\psi_3(x)-w_2(x). \end{aligned}$$

3. Uniqueness of solutions to the problem. We will prove the uniqueness of the problem under consideration by the method of energy integrals. In the domain of D_1 we have the equation $z_{xx}-z_y=0$

$$\iint_{D_1} z(z_{xx}-z_y)dxdy=\iint_{D_1}(zz_{xx}-zz_{yy})dxdy=0.$$

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Can express zz_{xx} via $zz_{xx} = \frac{\partial}{\partial x}(zz_x) - z_x^2$ then the last equality takes the form:

$$\iint_{D_1} \frac{\partial}{\partial x}(zz_{xx}) - z_x^2 - zz_y)dxdy = 0.$$

Applying Green's formula, we get the following:

$$-\int_0^1 zz_x dy - \iint_{D_1} z_x^2 dxdy = \iint_{D_1} zz_y dxdy$$

Let us show that the second integral of the left side of the equality is equal to zero. To do this, we use Green's formula and since

$$-\int_0^1 zz_x dy - \iint_{D_1} z_x^2 dxdy = 0 \tag{4}$$

we have

$$\int_0^1 z(0, y) z_x(0, y) dy \leq 0 \tag{5}$$

Integrating the identity

$$z\left[z_{xx} - (-x)^m z_{yy}\right] = \frac{\partial}{\partial x}(zz_{xx}) - \frac{\partial}{\partial y}\left[(-x)^m zz_y\right] - z_x^2 + (-x)^m z_y^2$$

by domain D_2 and applying Green's formula to the right side of the equality, we have

$$\int_{\partial D_2} z\left[z_x dy + (-x)^m z_y dx\right] - \iint_{D_2} \left[z_x^2 + (-x)^m z_y^2\right] dxdy = 0$$

Let us divide the first integral into three parts i.e. and integrating by parts, respectively, we have

$$\int_C^B z\left[z_x dy + (-x)^m z_y dx\right] + \int_C^O z\left[z_x dy + (-x)^m z_y dx\right] \geq 0.$$

Consequently

$$\int_0^1 z(0,y) z_x(0,y) dy \geq 0 \tag{6}$$

Then, inequalities (5) and (6) lead to the equality

$$\int_0^1 z(0,y) z_x(0,y) dy = 0.$$

Therefore, from (4) we obtain

$$\iint_{D_1} z_x^2 dxdy = 0,$$

means, $z(x,y) = \mu(y)$, from the boundary condition $z\big|_{AD} = 0$ follows $z(x,y) \equiv 0$ in D_1 and from $z\big|_{BD} = -\omega_1(x)$ we get $u(x,y) \equiv 0$ in D_1. Insofar as $z\big|_{OC} = 0$, $z\big|_{OB} = 0$ and from the uniqueness of the Cauchy problem in the hyperbolic domain we obtain $u(x,y) \equiv 0$ in D_2, which was to be proved.
Existence of a solution to the problem. It is known that the solution of the Cauchy problem for the equation $L_2 z = 0$ in the domain of D_2 has the form

$$z(\xi,\eta) = \int_0^{\xi} (\eta - \zeta)^{-\beta} (\xi - \zeta)^{-\beta} T(\zeta) d\zeta + \int_{\xi}^{\eta} (\eta - \zeta)^{-\beta} (\zeta - \xi)^{-\beta} N(\zeta) d\zeta \tag{7}$$

Where

$$N(\zeta) = \frac{1}{2\cos \pi\beta} T(\zeta) - \gamma_2 \nu(\zeta) \tag{8}$$

$$\tau(y) = z(0,y), \quad 0 \leq y \leq 1,$$

$$\nu(y) = \lim_{x \to -0} \frac{\partial u}{\partial x} = [2(1-2\beta)]^{-2\beta} \lim_{\eta - \xi \to -0} (\eta - \xi)^{2\beta} \left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta} \right),$$

Because $z(x,y)$ is generalized solution of the Cauchy problem for the equation $L_2 z = 0$ in the domain of D_2 from the class R_2 then has representation (7) and

$$\tau(y) = \tau(0) + \int_0^y (y - t)^{-2\beta} T(t) dt \tag{9}$$

and functions $T(t)$ and $\nu(t)$ are continuous and integrable on $(0, 1)$, where

$$\gamma_2 = [2(1-2\beta)]^{2\beta - 1} \frac{\Gamma(2-2\beta)}{\Gamma^2(1-\beta)}, \ \beta = \frac{m}{2(m+2)}.$$

To represent the solution of the equation $L_2 z = 0$ in the domain of D_2 satisfying the boundary conditions $z|_{OB} = \tau(y)$, $z|_{AD} = \psi(y)$, $z|_{OA} = \tau_1(x) - w_1(x)$ we use the solution of the first boundary value problem, i.e.

$$z(x,y) = \int_0^y \tau(\eta) G_\xi(x,y;0,\eta)d\eta + \\ + \int_0^1 [\tau_1(\xi) - w_1(\xi)] G(x,y;\xi,0)d\xi - \int_0^y \psi(\eta) G_\xi(x,y;1,\eta)d\eta, \quad (10)$$

Where $G(x,y;\xi,\eta)$ Green's function of the first boundary value problem for the heat equation and it has the form[7-8]:

$$G(x,y;\xi,\eta) = \sum_{n=-\infty}^{+\infty} [z(x,y;\xi + 2n) - z(x,y;-\xi + 2n,\eta)]$$

where

$$z(x,y;\xi,\eta) = \frac{1}{2\sqrt{\pi}} \begin{cases} \frac{1}{\sqrt{y-\eta}} e^{-\frac{(x-\xi)^2}{4(y-\eta)}} & at \;\; y > \eta \\ 0 & at \;\; y \le \eta \end{cases}$$

To define an unknown function $w_1(x)$ implement the condition

$$z|_{BD} = \psi_1(x) - w_1(x) \quad (11)$$

Based on (10), (11) and that in $BD: y = 1$ then we get

$$\psi_1 - w_1(x) = \int_0^1 \tau(\eta) G_\xi(x,1;0,\eta)d\eta + \int_0^1 \tau_1(\xi) G(x,1;\xi,0)d\xi - \\ - \int_0^1 w_1(\xi) G(x,1;\xi,0)d\xi - \int_0^1 \psi(\eta) G_\xi(x,1;1,\eta)d\xi$$

The last equality can be expressed as follows

$$w_1(x) - \int_0^1 w_1(\xi) G(x,1;\xi,0)d\xi = g(x) \quad (12)$$

where

$$g(x) = \psi_1(x) - \int_0^1 \tau(\eta) G_\xi(x,1;0,\eta)d\eta - \\ - \int_0^1 \tau(\xi) G(x,1;\xi,0)d\xi + \int_0^1 \psi(\eta) G_\xi(x,1;1,\eta)d\eta.$$

Equation (12) is an integral Fredholm equation of the second kind, the solvability of which follows from the uniqueness of the solution to the problem and is determined by the formula

$$w_1(x)=g(x)+\int_0^1 g(\xi)R(x,\xi;-1)d\xi$$

Calculating the derivative $\frac{\partial z}{\partial x}$, then letting x tend to zero, taking into account (9) and the Dirichlet transformation, we have

$$\begin{aligned}\nu(y)=&\frac{2\beta}{\sqrt{\pi}}\int_0^y T(t)dt\int_t^y (y-\eta)^{-\frac{1}{2}}(\eta-t)^{-2\beta-1}d\eta+\int_0^y T(t)dt\int_t^y K_1(y,\eta)(\eta-t)^{-2\beta}d\eta+\\&+\int_0^1 T(s)ds\int_0^1 G_x(0,y;t,0)dt\int_s^1(\eta-s)^{-2\beta}G_\xi(t,1;0,\eta)d\eta+\\&+\int_0^1 T(s)ds\int_0^1 G_x(0,y;z,0)dz\int_0^1 R(z,t;-1)dt\int_s^1 G_\xi(t,1;0,\eta)(\eta-s)^{-2\beta}d\eta+\Phi_2(y).\end{aligned}\tag{13}$$

Where

$$K_1(y,\eta)=G_{\xi x}(x,y;0,\eta)\Big|_{x=0}=\frac{1}{2\sqrt{\pi}}\left[\frac{1}{(y-\eta)^{\frac{3}{2}}}\sum_{\substack{n=-\infty\\n\neq 0}}^{+\infty}e^{\frac{n^2}{y-\eta}}-\frac{1}{(y-\eta)^{\frac{5}{2}}}\sum_{\substack{n=-\infty\\n\neq 0}}^{+\infty}n^2e^{-\frac{n^2}{y-\eta}}\right]$$

$$\begin{aligned}\Phi_2(y)=&-\int_0^1\Phi_1(\xi)G_x(0,y;\xi,0)d\xi+\int_0^1\tau_1(\xi)G_x(0,y;\xi,0)d\xi-\\&-\int_0^y\psi(\eta)G_{\xi x}(0,y;1,\eta)d\eta.\end{aligned}$$

We extend the first and second integrals on the right side of (13) with respect to ***t*** to$(0,1)$those.

$$\int_0^1 K_2(y,t)T(t)dt=\frac{2\beta}{\sqrt{\pi}}\int_0^y T(t)dt\int_0^y(y-\eta)^{-\frac{1}{2}}(\eta-t)^{-2\beta-1}d\eta$$

$$\int_0^1 K_3(y,t)T(t)dt\equiv\int_0^y T(t)dt\int_t^y K_1(y,\eta)(\eta-t)^{-2\beta}d\eta$$

Where

$$K_2(y,t)=\begin{cases}\frac{2\beta}{\sqrt{\pi}}\int_t^y(y-\eta)^{-\frac{1}{2}}(\eta-t)^{-2\beta-1}d\eta & at\ \ 0\le t\le y,\\ 0 & at\ \ y<t\le 1,\end{cases}$$

$$K_3(y,t)=\begin{cases}\int_t^y (y-t)^{-2\beta} K_1(y,\eta)\,d\eta & at\ \ 0\le t\le y,\\ 0 & at\ \ y<t\le 1.\end{cases}$$

Then (13) has the form

$$\nu(y)=\int_0^1 K(y,t)T(t)dt+\Phi_2(y) \qquad (14)$$

where

$$K(y,t)=K_2(y,t)+K_3(y,t)+\int_0^1 G_x(0,y;t,0)dt\cdot\int_s^1(\eta-s)^{-2\beta}G_\xi(t,1;0,\eta)d\eta+$$
$$+\int_0^1 G_x(o,y;z,0)dz\int_0^1 R(z,t;-1)dt\int_s^1 G_\xi(t,1;0,\eta)(\eta-s)^{-2\beta}d\eta.$$

From (8) we find

$$-\nu(y)=\frac{1}{\gamma_2}\left[N(y)-\frac{1}{2\cos\pi\beta}T(y)\right]. \qquad (15)$$

Taking into account the gluing condition and excluding $z_x(0,y)=\nu(y)$ from (14) and (15), we have

$$N(y)-\frac{1}{2\cos\pi\beta}T(y)=\gamma_2\int_0^1 K(y,t)T(t)dt+\gamma_2\Phi_2(y)$$

or

$$T(y)+2\gamma_2\cos\pi\beta\int_0^1 K(y,t)T(t)dt=2\cos\pi\beta N(y)-2\gamma_2\cos\pi\beta\,\Phi_2(y). \quad (16)$$

The study of equation (16) shows that it is an integral Fredholm equation of the second kind with a weak singularity. Its unique solvability follows from the uniqueness of the solution of the problem. Solutions of the integral equation (16) can be written using the resolvent as

$$T(y)=2\cos\pi\beta[N(y)-\gamma_2\Phi_2(y)]-4\gamma_2\cos^2\pi\beta\times\int_0^1 R_1(y,s;\lambda)[N(s)-\gamma_2\Phi_2(s)]ds, \qquad 17)$$

Where $R_1(y,s;\lambda)$ is resolvent of equation (16).

Subordinating (7) to the conditions on the characteristics of OC, BC $z|_{OC}=\psi_2(x),\quad z|_{BC}=\psi_3(x)-w_2(x).$ and taking into account (17), i.e. on the $OC:\xi=0$ from (7) and denoting $x=-[2(1-2\beta)]^{2\beta-1}\eta^{1-2\beta}$ we get

$$\psi_2\left\{-\left[2(1-2\beta)\right]^{2\beta-1}\eta^{1-2\beta}\right\}=\int_0^{\eta}(\eta-\zeta)^{-\beta}\zeta^{-\beta}N(\zeta)d\zeta \quad (18)$$

Based $BC:\eta=1$ and $x=-[2(1-2\beta)]^{2\beta-1}(1-\xi)^{1-2\beta}$ we get

$$w_2\left\{-\left[2(1-2\beta)\right]^{2\beta-1}(1-\xi)^{1-2\beta}\right\}+\int_0^1 K_4(\xi,\zeta)(1-\zeta)^{-\beta}|\xi-\zeta|^{-\beta}d\zeta=F(\xi) \quad (19)$$

where

$$K_4(\xi,\zeta)=\begin{cases}2\cos\pi\beta N(\zeta)-4\gamma_2\cos^2\pi\beta\int_0^1 N(s)R_1(\zeta,s;\lambda)ds & at\ 0\le\zeta\le\xi,\\ N(\zeta) & at\ \xi<\zeta\le 1\end{cases}$$

$$F(\xi)=\psi_3\left\{-\left[2(1-2\beta)\right]^{2\beta-1}(1-\xi)^{1-2\beta}\right\}-4\gamma_2^2\cos^2\pi\beta\int_0^{\xi}(1-\zeta)^{-\beta}(\xi-\zeta)^{-\beta}\times$$

$$\times\int_0^1\Phi_2(s)R_1(\zeta,s;\lambda)dsd\zeta+2\gamma_2\cos\pi\beta\int_0^{\xi}(1-\zeta)^{-\beta}(\xi-\zeta)^{-\beta}\Phi_2(\zeta)d\zeta.$$

The latter system has a solution, which proves the existence of a solution to the Dirichlet problem.

Studies on the smoothness of given functions. It can be seen that if we use from (18), we can find $T(y)$ those. using the fractional operator, we rewrite $N(\eta)$ in the following form:

$$N(\eta)=\frac{\eta^{\beta}}{\Gamma(1-\beta)}D_{0\eta}^{1-\beta}\psi_2\left\{-\left[2(1-2\beta)\right]^{2\beta-1}\eta^{1-2\beta}\right\}$$

Therefore, from (19) $w_2\left\{-\left[2(1-2\beta)\right]^{2\beta-1}(1-\xi)^{1-2\beta}\right\}$ takes the form:

$$\begin{aligned}&w_2\left\{-\left[2(1-2\beta)\right]^{2\beta-1}(1-\xi)^{1-2\beta}\right\}=\psi_3\left\{-\left[2(1-2\beta)\right]^{2\beta-1}(1-\xi)^{1-2\beta}\right\}-\\&-\frac{2\cos\pi\beta}{\Gamma(1-\beta)}\int_0^{\xi}(1-\zeta)^{-\beta}(\xi-\zeta)^{-\beta}\zeta^{\beta}D_{0\zeta}^{1-\beta}\psi_2\left\{-\left[2(1-2\beta)\right]^{2\beta-1}\zeta^{1-2\beta}\right\}d\zeta-\\&-\frac{1}{\Gamma(1-\beta)}\int_{\xi}^{1}(1-\zeta)^{-\beta}(\xi-\zeta)^{-\beta}\zeta^{\beta}D_{0\zeta}^{1-\beta}\psi_2\left\{-\left[2(1-2\beta)\right]^{2\beta-1}\zeta^{1-2\beta}\right\}d\zeta-\\&-4\gamma_2^2\cos^2\pi\beta\cdot J_1+2\gamma_2\cos\pi\beta\cdot J_2+\frac{4\gamma_2^2\cos^2\pi\beta}{\Gamma(1-\beta)}\cdot J_3\end{aligned} \quad (20)$$

where

$$J_1=\int_0^{\xi}(1-\zeta)^{-\beta}(\xi-\zeta)^{-\beta}\int_0^1\Phi_2(s)R_1(\zeta,s;\lambda)dsd\zeta \quad (21)$$

$$J_2=\int_0^{\xi}(1-\zeta)^{-\beta}(\xi-\zeta)^{-\beta}\Phi_2(\zeta)\,d\zeta$$

$$J_3=\int_0^{\xi}(1-\zeta)^{-\beta}(\xi-\zeta)^{-\beta}\int_0^1 R_1(\zeta,s;\lambda)s^{\beta}D_{0s}^{1-\beta}\psi_2\left\{-\left[2(1-2\beta)\right]^{2\beta-1}s^{1-2\beta}\right\}ds\,d\zeta \quad (22)$$

$$\Phi_2(y)=-\int_0^1[\psi_1(\xi)-\int_0^1\tau_1(s)G(\xi,1;s,0)ds+\int_0^1\psi(\eta)G_{\xi}(\xi,1;1,\eta)d\eta+$$

$$+\int_0^1\psi_1(s)R(\xi,s;-1)ds-\int_0^1\int_0^1\tau_1(t)G(\xi,1;t,0)dtR(\xi,s;-1)ds+$$

$$+\int_0^1\int_0^1\psi(\eta)G_{\xi}(t,1;1,\eta)d\eta R(\xi,t;-1)dt]G_x(0,y;\xi,0)d\xi+$$

$$+\int_0^1\tau_1(\xi)G_x(0,y;\xi,0)d\xi-\int_0^y\psi(\eta)G_{\xi x}(0,y;1,\eta)d\eta.$$

Let us present some auxiliary expansions of the Green's function involved inside the integral as a kernel

$$G_{\xi}(\xi,1;1,\eta)=\sum_{n=-\infty}^{+\infty}\left[\frac{(\xi-1-2n)}{(1-\eta)^{\frac{3}{2}}}e^{-\frac{(\xi-1-2n)^2}{4(1-\eta)}}-\frac{(\xi+1-2n)}{2(1-\eta)^{\frac{3}{2}}}e^{-\frac{(\xi+1-2n)^2}{4(1-\eta)}}\right],$$

$$G(\xi,1;t,0)=\sum_{n=-\infty}^{+\infty}\left[e^{-\frac{(\xi-t-2n)^2}{4}}-e^{-\frac{(\xi+t-2n)^2}{4}}\right],$$

$$G_x(0,y;\xi,0)=\sum_{n=-\infty}^{+\infty}\left[\frac{\xi+2n}{2y^{\frac{3}{2}}}e^{-\frac{(\xi+2n)^2}{4y}}-\frac{\xi-2n}{2y^{\frac{3}{2}}}e^{-\frac{(\xi-2n)^2}{4y}}\right],$$

$$G_{\xi x}(0,y;1,\eta)=\sum_{n=-\infty}^{+\infty}\left[\frac{(-1-2n)^2}{4(y-\eta)^{\frac{5}{2}}}e^{-\frac{(-1-2n)^2}{4(y-\eta)}}-\frac{(1-2n)^2}{4(y-\eta)^{\frac{5}{2}}}e^{-\frac{(1-2n)^2}{4(y-\eta)}}\right]$$

For (21) to take place, it is necessary$\Phi_2$was a continuous function, then from the representation$\Phi_2(y)$it easily follows that ψ_1, τ_1 continuous. Hence from $\int_0^1\psi(\eta)G_{\xi}(\xi,1;1,\eta)d\eta$ function ψ should look like $\psi(\eta)=(1-\eta)^{\frac{3}{2}}\psi^*(\eta)$ where ψ^* is a continuous function. Now from (22) we will study the function ψ_2. From the definition of integro-differential operators of fractional order $\alpha>0$ those. From

$D_{ax}^{\alpha} f(x) = \frac{d^n}{dx^n}\{D_{ax}^{-(n-\alpha)} f(x)\}$ where $n-1<\alpha<n,\ \ n\geq 1$, $f(x)\in L(a;b)$ and because $m \in (-\frac{16}{9};-\frac{7}{4})$, $\beta\in(-4;-3.5)$, follows that $4<1-\beta<5 \Rightarrow n=5$ and

$$D_{0s}^{1-\beta}\psi_2\left\{-\left[2(1-2\beta)\right]^{2\beta-1} s^{1-2\beta}\right\} = \frac{d^5}{ds^5}\left[D_{as}^{-(4+\beta)}\psi_2\left\{-\left[2(1-2\beta)\right]^{2\beta-1} s^{1-2\beta}\right\}\right]$$

$$\frac{d^5}{ds^5}\left[D_{as}^{-(4+\beta)}\psi_2\left\{-\left[2(1-2\beta)\right]^{2\beta-1} s^{1-2\beta}\right\}\right] = = \frac{d^5}{dx^5}\left[\frac{1}{\Gamma(1-\beta)}\int_0^s (s-\eta)^{3+\beta}\psi_2\left\{-\left[2(1-2\beta)\right]^{2\beta-1}\eta^{1-2\beta}\right\}d\eta\right] \tag{23}$$

It can be seen that the right side of equation (23) has a weak feature. Therefore, we cannot immediately differentiate it. To avoid this, we will first integrate by parts and then differentiate. Repeating this process five times and putting the result obtained in (22), then we choose among them the term that has the largest singularity, i.e. $\int_0^s (s-\eta)^{3+\beta} s^{\beta}\psi_2^{(5)}\left\{-\left[2(1-2\beta)\right]^{2\beta-1}\eta^{1-2\beta}\right\}d\eta$. The study of this expression shows that the existence of the integral depends on the continuity of the function belonging to the kernel. To do this, we will do the following:

$$\psi_2\left\{-\left[2(1-2\beta)\right]^{2\beta-1}\eta^{1-2\beta}\right\} = \eta^{3-2\beta}\psi_2^*\left\{-\left[2(1-2\beta)\right]^{2\beta-1}\eta^{1-2\beta}\right\}$$

In sofar as $x = -[2(1-2\beta)]^{2\beta-1}\eta^{1-2\beta} \Rightarrow \eta = kx^{\frac{1}{1-2\beta}}$ where $k = \frac{(-1)^{\frac{1}{1-2\beta}}}{2(1-2\beta)}$ then we get

$\psi_2(x) = k^{3-2\beta} x^{\frac{3-2\beta}{1-2\beta}}\psi_2^*(x)$. From (20) it can be seen that the function $\psi_3(x)\in C^2(\overline{D}_2)$.

Based on the above results, we will formulate the following theorem:

Theorem. If $\psi_3(x)\in C^2(\overline{D}_2)$, ψ_1, τ_1 continuous functions and $\psi(x)$, $\psi_2(x)$ represent in the form $\psi(x) = (1-x)^{\frac{3}{2}}\psi^*(x)$ and $\psi_2(x) = k^{3-2\beta}x^{\frac{3-2\beta}{1-2\beta}}\psi_2^*(x)$, where ψ^*, ψ_2^* continuous functions, then the solution of the Dirichlet problem exists and is unique.

3 Conclusions

Thus, with the help of energy integrals, the uniqueness of the solution of the boundary value problem for the homogeneous equation of parabolic - hyperbolic type of the third order of the second kind is proved. Necessary and sufficient conditions for the existence of a generalized solution of the formulated problem are found. An explicit representation of the solution of the problem under study is obtained. The results obtained and the developed methods make it possible to further investigate similar boundary value problems for a homogeneous parabolic-hyperbolic type equation of the third order of the second kind.

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