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Free boundary problem for predator-prey model

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Abstract. In this article, we study the behavior of two species evolving in a domain with a free boundary. This system mimics the spread of invasive or new predator species, in which free boundaries represent the expanding fronts of predator species and are described by the Stefan condition. A priori estimates for the required functions are established. These estimates are used to prove the existence and uniqueness of the solution.

1 Introduction

Today, humanity faces increasingly serious environmental and epidemiological problems, such as environmental pollution, the invasion of exotic species, the emergence of new infectious diseases, and the resumption of existing epidemiological diseases. Recently, Mathematical modeling has been successfully used to study many biomedical and epidemiological problems, and in all these contexts, nonlinear and complex dynamics have been observed [1-7]. Over the past twenty years, significant progress has been made in the mathematical modeling of biomedical processes, leading to more complex models consisting of nonlinear partial differential equations systems. Population dynamics is one of the most widely discussed topics in biomathematics. The study of the evolution of different populations has always been of particular interest, starting with populations of the same species but gradually moving to more realistic models in which different species live and interact in the same habitat. Among them, we can find models that study competitive relationships, symbiosis, commensalism, or predator-prey relationships.

Studying the spatial and temporal behavior of predator and prey in an ecological system is important in population ecology. To study the predator-prey system, various types of mathematical models have been proposed [10, 11]. These studies provide a theoretical framework for understanding the complex spatiotemporal dynamics observed in real ecological systems. Such models are mathematically interesting, and a rigorous mathematical analysis of these models, such as global existence, uniqueness, and stability of solutions, is attracting more and more attention.

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In this article, we consider the following model:

$$\frac{\partial u}{\partial t} = d_1 u_{xx} + m_1 u_x + u(1-u) - \frac{vu}{u+m}, \quad (t,x) \in D, \tag{1}$$

$$\frac{\partial v}{\partial t} = d_2 v_{xx} + m_2 v_x + kv(1 - \frac{bv}{u+a}), \quad (t,x) \in D, \tag{2}$$

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad 0 \le x \le s_0 = s(0), \tag{3}$$

$$u_x(t,0) = u(t,s(t)) = 0, \qquad 0 \le t \le T, \tag{4}$$

$$v_x(t,0) = v(t,s(t)) = 0, \qquad 0 \le t \le T, \tag{5}$$

$$\dot{s}(t) = -\mu\big(u_x(t,s(t)) + \rho v_x(t,s(t))\big), \qquad 0 \le t \le T, \tag{6}$$

where $D = \{(t,x) : t > 0, 0 < x < s(t)\}$, $u(t,x)$, $v(t,x)$ denote the population densities of the two competitors, and all parameters are positive numbers.

From a biological point of view, model (1)-(6) describes how the two species evolve if they initially occupy the bounded region $[0, s_0]$. The homogeneous Neumann boundary condition at $x = 0$ indicates that the left boundary is fixed, with the population confined to move only to the right of the boundary point $x = 0$. We assume both species tend to emigrate through the right boundary point to obtain their new habitat: the free boundary $x = s(t)$ represents the spreading front. Moreover, it is assumed that the expanding speed of the free boundary is proportional to the normalized population gradient at the free boundary. This is well-known as the Stefan condition.

Regarding the problem data, the following conditions are assumed to be satisfied:

i. Parameters a, b, μ, m, k, ρ, m_i and d_i ($i = 1.2$) are positive constants;

ii. $u_0(x)$, $v_0(x)$ satisfy the following conditions:

$$u_0(x) \in C^{2+\alpha}[0,L], \; u_0(x) > 0 \text{ to } 0 \le x \le s_0, \quad u_0(0) = u_0(s_0) = 0;$$

$$v_0(x) \in C^{2+\alpha}[0,s_0], \; v_0(x) \ge 0 \text{ to } 0 \le x \le s_0, v_0(0) = v_0(s_0) = 0.$$

When $m_1 = m_2 = 0$, the problem (1)-(6) was investigated in the works [13-15] and proved a spreading-vanishing dichotomy. In [1,16,17], the authors studied the free boundary problem for a reaction-diffusion system with a linear convection term. They obtained a dichotomy result and presented a constant asymptotic spreading speed of the expanding front.

The rest of the paper is organized as follows. First, we establish two-sided bounds for $u(t,x)$, $v(t;x)$ and $\dot{s}(t)$, and then a Holder norm bounds for $u(t;x)$, $v(t;x)$.

2 A priori estimates and global existence

First, we establish some a priori estimates for the problem (1)-(6).

Lemma 1. - Let $(u(t,x), v(t,x), s(t))$ be a solution of problem (1)-(6) for $t\in[0,T]$. Then we have the following estimates

$$0 < u(t,x) \le \max\left\{1, \| u_0 \|_\infty\right\} = M_1, \quad (t,x)\in \bar{Q}, \tag{7}$$

$$0 < v(t,x) \le \max\left\{M_1 + a, \| v_0 \|_\infty\right\} = M_2, \quad (t,x)\in \bar{D}, \tag{8}$$

$$0 < \dot{s}(t) \le \mu N_1 + \mu\rho N_2 = M_3, \quad 0 < t \le T, \tag{9}$$

where $N_1 = \max\limits_{x\in[0,s_0]}\left\{\dfrac{M_1}{m_1}, \dfrac{u_0(x)}{s_0 - x}\right\}$, $N_2 = \max\limits_{x\in[0,s_0]}\left\{\dfrac{M_2 k}{m_2}, \dfrac{v_0(x)}{s_0 - x}\right\}$

Proof: first we prove the positivity of the function $v(t,x)$. Take an arbitrary point $P\in D$ such that $v(P)=0$. At this point, the right-hand side of (2) should be zero. And also, at this point, the function $v(t,x)$ reaches its minimum value. Hence, according to the usual maximum principle $v(P)=v(P_0)$ for all $P_0 \in D_0 = (t_0, x): 0 < t_0 \le T, 0 < x < s(t_0)$ and we arrive at a contradiction. The resulting contradiction proves that $v(t,x) > 0$ in D.

Similarly, we have $u(t,x)>0$ for $\overline{D}$, since $u(t,s(t))=0,$ the Hopf lemma, then $u_x(t,s(t))<0$ for all $t\in(0,T]$. $v(t,x)>0$ for $\bar{D}$, since $v(t,s(t))=0,$ the Hopf lemma then $v_x(t,s(t))<0$ for all $t\in(0,T]$. It follows from the free boundary condition in (6) that $\dot{s}(t)>0$ for $t\in(0,T]$.

It follows from the comparison principle that

$$u(t,x) \le \bar{u}(t,x) \le \bar{u}(t) \le \max\left\{1, \| u_0 \|_\infty\right\} = M_1 \text{ in } \bar{D},$$

where $\overline{u}(t)$ the solution of the problem

$$\begin{cases} \bar{u}'(t) = \bar{u}(1-\bar{u}), & t>0, \\ \bar{u}(0) = \| u_0 \|_\infty . \end{cases}$$

Similarly, considering the following initial value problem

$$\begin{cases} \bar{v}'(t) = k\bar{v}\left(1 - \dfrac{\bar{v}}{M_1 + a}\right), & t>0, \\ \bar{v}(0) = \| v_0 \|_\infty , \end{cases}$$

by comparison principle, we obtain that

$$v(t,x) \le \bar{v}(t) \le \max\left\{M_1 + a, \| v_0 \|_\infty\right\} = M_2 \text{ in } \bar{D}.$$

To derive an upper bound for $\dot{s}(t)$, to this end, we compare $u(t,x)$, $v(t,x)$ to the auxiliary functions $U(t,x)$, $V(t,x)$ defined by

$$\begin{aligned} U(t,x) &= u(t,x) + N_1 \cdot (x - s(t)), \\ V(t,x) &= v(t,x) + N_2 \cdot (x - s(t)), \end{aligned} \qquad N_1,\ N_2 - const. \tag{10}$$

We find that

$$\begin{cases} U_t - d_1 U_{xx} - c_1 U_x \le M_1 - m_1 N_1 \le 0 \quad \text{in} \quad \bar{D}, \\ V_t - d_2 V_{xx} - c_2 V_x \le M_2 k - m_2 N_2 \le 0 \quad \text{in} \quad \bar{D}, \\ U_x(t,0) = N_1 > 0,\ V_x(t,0) = N_2 > 0, \quad 0 \le t \le T, \\ U(0,x) = u_0(x) + N_1(x - s_0) \le 0, \quad 0 \le x \le s_0, \\ V(0,x) = v_0(x) + N_2(x - s_0) \le 0, \quad 0 \le x \le s_0, \\ U(t,s(t)) = V(t,s(t)) = 0. \end{cases} \tag{11}$$

By applying the maximum principle one more time(11), we obtain

$$U(t,x) \le 0, \qquad V(t,x) \le 0, \quad (t,x) \in D.$$

Then, (10) also implies that

$$u(t,x) \le N_1(s(t) - x), \quad v(t,x) \le N_2(s(t) - x), \quad 0 \le x \le s(t).$$

Therefore, $U_x(t,s(t)) = u_x(t,s(t)) + N_1 > 0,$

$$V_x(t,s(t)) = v_x(t,s(t)) + N_2 > 0,$$

or $u_x(t,s(t)) \ge -N_1$, $v_x(t,s(t)) \ge -N_2$,

Then we get (9), which completes the proof.

We will establish Holder norm bounds $|\cdot|_{1+\alpha}$ and $|\cdot|_{2+\alpha}$ in $\bar{D}$.

The boundary conditions of the problem (1) - (6) do not allow us to use the well-known results of [8]. Therefore, firstly, we introduce a transformation to straighten the free boundary

$$t = t, \quad y = \frac{x}{s(t)}.$$

For each equation of the system, we separately formulate the corresponding problem: Then, the domain D corresponds to the domain $Q = \{(t,y): 0 < t < T, 0 < y < 1\}$, and the bounded functions $U(t,y) = u(t,x)$, $V(t,y) = v(t,x)$ are a solution to the problem

$$\begin{cases} U_t - A_1\left(t,s(t)\right)U_{yy} - B_1(t,y,s(t),\dot{s}(t),U,V,U_y) = 0, & (t,y)\in Q, \\ U\left(0,s_0y\right) = u_0(s_0y), & 0\le y\le 1, \\ U\left(t,1\right) = 0, & 0\le t\le T, \\ U_y\left(t,0\right) = 0, & 0\le t\le T, \end{cases} \quad (12)$$

$$\begin{cases} V_t - A_2\left(t,s(t)\right)V_{yy} - B_2\left(t,y,s(t),\dot{s}(t),U,V,V_y\right) = 0, & (t,y)\in Q, \\ V\left(0,s_0y\right) = V_0(s_0y), & 0\le y\le 1, \\ V\left(t,1\right) = 0, & 0\le t\le T, \\ V\left(t,0\right) = 0, & 0\le t\le T, \end{cases} \quad (13)$$

where $A_1(t,s(t)) = \dfrac{4d_1s_0^2}{s^2(t)}$, $A_2(t,s(t)) = \dfrac{4d_2s_0^2}{s^2(t)}$,

$$B_1(t,y,s(t),\dot{s}(t),U,V,U_y) = \frac{y\dot{s}(t)+m_1}{s(t)}U_y - U(1-U) - V\left(\frac{U}{U+m}\right),$$

$$B_2(t,y,s(t),\dot{s}(t),U,V,U_y) = \frac{y\dot{s}(t)+m_2}{s(t)}V_y + kV\left(1-\frac{bV}{U+a}\right).$$

Conditions for unknown boundaries will take the form:

$$\dot{s}(t) = -\frac{2\mu}{s(t)}U_y(t,1) - \frac{2\mu\rho}{s(t)}V_y(t,1), \quad t>0. \quad (14)$$

For all equations in problems (12) and (13), the parabolicity condition and the subordination condition for lower-order terms are satisfied (see [8]), which allows you to directly apply the results of [8].

We formulate the theorem for the function $V\left(t,y\right)$.

Similar results are valid for $U\left(t,y\right)$.

Theorem 2. - Let the function $V\left(t,y\right)$, $M_2 = \max_{\bar{Q}} |V(t,y)|$ continuous in $\bar{Q}$ together with V_y and satisfies the conditions of the problem (13). Then

$$\left|V_y\left(t,y\right)\right| \le C_1\left(M_2\right), \quad \left(t,y\right)\in\bar{Q}.$$

And if it is also known that the function $V\left(t,y\right)$ has in Q summable with a square generalized derivatives V_{yy} and V_{ty}, then there exists $\alpha = \alpha(M_2)$ that

$$\left|V\left(y,\tau\right)\right|_{1+\alpha}^{\bar{Q}_2} \le C_2\left(M_2,C_1\right).$$

Let the function $V\left(t,x\right)$ satisfying the equation (13) in Q, continuous in Q at the place with derivatives V_t, V_y, V_{yy} and $\left|V\left(t,y\right)\right|_{2+\alpha}^{\bar{Q}} < \infty$. Then

$$\left|V\left(t,y\right)\right|_{2+\alpha}^{\bar{Q}} \le C_3(M_2,C_1,C_2)$$

where $A_{20} = \min\limits_{\bar{Q}} A_2$, $\Gamma\left(t=0, y=0, y=1\right)$ - parabolic boundary.

The proof is given as in Theorems 3 and 4 in [8].

3 Existence and uniqueness of solutions

In this section, we first state a result about the local existence and uniqueness of a solution to (1)-(6).

Theorem 2. Assume that (u_0, v_0) satisfies the condition (*ii*), then for any $\alpha \in (0,1)$, there is a $T > 0$ such that the problem (1)-(6) admits a unique solution ($u(t,x)$, $v(t,x)$, $h(t)$), which satisfies

$$u(t,x),\ v(t,x),\ h(t) \in C^{\frac{1+\alpha}{2},1+\alpha}(D) \times C^{\frac{1+\alpha}{2},1+\alpha}(D) \times C^{1+\frac{\alpha}{2}}([0,T]).$$

Proof. Then the, problem (1)-(6) is transformed into the following problems (12)-(13) with a fixed boundary. As mentioned above, we will use the contraction mapping principle to prove the local existence of a solution. We denote by $\bar{s} = -\mu\left(U'_0(1) + \rho V'_0\left(1\right)\right)$. As in [10] we shall prove the local existence

by using the contraction mapping theorem. We let T such that $0 < T \le \dfrac{s_0}{4(2+s_0)}$ and

introduce the function spaces

$$X_{1T} = \left\{U \in C\left(Q\right) : U\left(0,y\right) = U_0\left(y\right), \left\|U - U_0\right\|_{C(Q)} \le 1\right\},$$
$$X_{2T} = \left\{V \in C\left(Q\right) : V\left(0,y\right) = V_0\left(y\right), \left\|V - V_0\right\|_{C(Q)} \le 1\right\},$$
$$X_{3T} = \left\{s\left(t\right) \in C^1\left(\left[0,T\right]\right), \left\|s' - \bar{s}\right\|_{C([0,T])} \le 1\right\}.$$

where $Q = \{(t,x) : 0 \le t \le T,\ 0 < x < s_0\}$.

Then $X_T = \left\{X_{1T} \times X_{2T} \times X_{3T}\right\}$ is a complete metric space with metric

$$\mathrm{D}\left(\left(U_1,V_1,s_1\right),\left(U_2,V_2,s_2\right)\right) = \left\|U_1 - U_2\right\|_{C(Q_u)} + \left\|V_1 - V_2\right\|_{C(Q_v)} + \left\|s_1' - s_2'\right\|_{C([0,T])}.$$

Then we have

$$s(t) - s_0| \le \int_0^T s'(\eta)|\ d\eta \le T(s_0 + 2) \le \frac{s_0}{4},$$

and this ensures that the mapping $(t,x)\to(t,y)$ is a diffeomorphism.

As mentioned above, we will construct a contraction mapping from X_T into X_T to prove the existence of a local solution. We begin this construction now. As $0\le t\le T$, the coefficients A_1 and B_1 are bounded and A_1^2 is between two positive constants. By standard L_p theory and the Sobolev embedding theorem, for any $(U,V,s(t))\in X_T$, the following initial-boundary value problem arises:

$$\begin{cases} A_1(t,s)\bar{U}_{yy}+B_1(t,y,s,U,V,U_y)-\bar{U}_t=0, & t>0 \quad \text{and} \quad 0<y<1, \\ A_2(t,s)\bar{V}_{xx}+B_2(t,y,s,U,V,V_y)-\bar{V}_t=0, & t>0 \quad \text{and} \quad 0<y<1, \\ \bar{V}(0,y)=V_0(y), \quad 0,, y,, 1, \\ \bar{U}(0,y)=U_0(y), \quad 0,, y\le 1, \\ \bar{U}_y(t,0)=\bar{U}(t,1)=0,\ 0,, t,, T, \\ \overline{V_y}(t,0)=\bar{V}(t,1)=0,\ 0,, t,, T, \end{cases}$$

for any $\alpha\in(0,1)$, admits a unique bounded solution $(\bar{U},\bar{V})\in C^{1+\alpha,\frac{1+\alpha}{2}}(Q)\times C^{1+\alpha,\frac{1+\alpha}{2}}(Q)$. Moreover,

$$\|\bar{U}\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(Q_u)}\le C_1 \text{ and } \|\bar{V}\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(Q_v)}\le C_2,$$

where C_1 and C_2 depend on s_0, α, $\|U_0\|_{C^2[0,1])}$ and $\|V_0\|_{C^2[0,1]}$.

Next, we define

$$\bar{s}(t)=s_0-\mu\int_0^T\left[\bar{U}_y(\eta,1)+\rho\bar{V}_y(\eta,1)\right]d\eta.$$

Then $\bar{s}'(t)=-\mu\left(\bar{U}_y(\eta,1)+\rho\bar{V}_y(\eta,1)\right)\in C^{\frac{\alpha}{2}}[0,T]$ 0 $\|\bar{s}'\|_{C^{\frac{\alpha}{2}}}\le C_3$ and $\|\bar{s}'\|_{C^{\frac{\alpha}{2}}}\le C_3$, where C_3 depends on μ, s_0, α, $\|U_0\|_{C^2[0,1]}$ and $\|V_0\|_{C^2[0,1]}$.

We are now ready to consider the mapping defined on X_T by the formula:

$$\mathrm{F}:(U,V,s)\to(\bar{U},\bar{V},\bar{s},)$$

to find a fixed point. We first confirm that for sufficiently small T, F maps X_T to itself: indeed, if we take T such that

$$0 < T^* \le \min\left\{ C_1^{\frac{-2}{1+\alpha}}, C_2^{\frac{-2}{1+\alpha}}, C_3^{\frac{-2}{\alpha}} \right\},$$

then we have

$$\left\| \overline{s}' - s_1 \right\|_{C([0,T])} = \max_{t\in[0,T]} \left| \overline{s}' - s_1 \right| \le \max_{t\in[0,T], t\ne\tau} \left\{ \frac{\left| \overline{s}'(t) - s'(\tau) \right|}{\left| t-\tau \right|^{\frac{\alpha}{2}}} \right\} T^{\frac{\alpha}{2}} \le$$

$$\max_{t\in[0,T]} \left\| \overline{s}' \right\|_{C^{\frac{\alpha}{2}}} T^{\frac{\alpha}{2}} \le C_3 T^{\frac{\alpha}{2}} \le 1,$$

$$\left\| \overline{U} - U_0 \right\|_{C(Q)} = \max_{t\in[0,T], y\in[0,1]} \left| \overline{U} - U_0 \right| \le \max_{t\in[0,T], y\in[0,1]} \left\{ \frac{\left| \overline{U}(t,y) - U(0,y) \right|}{t^{\frac{1+\alpha}{2}}} t^{\frac{1+\alpha}{2}} \right\}$$

$$\le \max_{t\in[0,T], y\in[0,1], t\ne\tau} \left\{ \frac{\left| \overline{U}(t,y) - U(\tau,y) \right|}{\left| t-\tau \right|^{\frac{1+\alpha}{2}}} \right\} T^{\frac{1+\alpha}{2}} \le \left\| \overline{U} \right\|_{C^{\frac{1+\alpha}{2},0}} T^{\frac{1+\alpha}{2}} \le C_1 T^{\frac{1+\alpha}{2}} \le 1,$$

$$\left\| \overline{V} - V_0 \right\|_{C(Q)} = \max_{t\in[0,T], y\in[0,1]} \left| \overline{V} - V_0 \right| \le \max_{t\in[0,T], y\in[0,1]} \left\{ \frac{\left| \overline{V}(t,y) - V(0,y) \right|}{t^{\frac{1+\alpha}{2}}} t^{\frac{1+\alpha}{2}} \right\}$$

$$\le \max_{t\in[0,T], y\in[0,1], t\ne\tau} \left\{ \frac{\left| \overline{V}(t,y) - V(\tau,y) \right|}{\left| t-\tau \right|^{\frac{1+\alpha}{2}}} \right\} T^{\frac{1+\alpha}{2}} \le \left\| \overline{V} \right\|_{C^{\frac{1+\alpha}{2},0}} T^{\frac{1+\alpha}{2}} \le C_2 T^{\frac{1+\alpha}{2}} \le 1.$$

In other words, F maps X_T to X_T. Now we check that F is a contraction for a sufficiently small T.

It follows from the estimates above that the image of F lies in the compact subset X_T, and the standard argument also shows that F is continuous. Using the Schauder fixed point theorem, we conclude that F has a fixed point in X_T.

In what follows, we can use the Schauder bounds to obtain additional regularity a solution such as the Holder continuity for $\dot{\overline{s}}(t)$, and the second spatial derivatives functions $\overline{U}$ and $\overline{V}$.

4 Conclusions

The theory of parabolic equations of the reaction-diffusion type plays an important role in solving practical problems. Especially for reaction-diffusion-type parabolic equations, free boundary value problems are widely used in mathematical physics, mechanics, and technology, especially in nanotechnology.

The main contribution of this article is to determine the global existence of the classical solution of problem (1)-(6) and to study the behavior of the solution. For a new class of free boundary value problems for mixed two-phase equations, a method for setting a priori predictions of the Shauder type was proposed. The principle of comparison was proven. The problem of the existence and uniqueness of the solution of the free boundary value problem (1)-(6) was studied.

The results obtained in this study allow the study of free boundary value problems for a parabolic equation of the reaction-diffusion type in the future. Hopefully, our work will encourage the study of various free boundary value problems for many parabolic equations.

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