Limited different schemes for mutual diffusion problems

D. K. Muhamediyeva[1,2*], *S.* Muminov[3], and *M. E.* Shaazizova[1], *Ch.* Hidirova[1] and *Yu.* Bahromova[1]

[1]Tashkent University of Information Technologies named after Muhammad al-Khwarizmi, Tashkent, Uzbekistan
[2]"Tashkent Institute of Irrigation and Agricultural Mechanization Engineers" National research University, Tashkent, Uzbekistan
[3]Urgench State University, Urgench, Uzbekistan

Abstract. In the article, the problems of mutual diffusion, represented by a system of nonlinear differential equations of parabolic type, were modeled following numerical solving methods. The developed software was tested using various examples and obtained positive results. Mutual diffusion processes were visualized.

1 Introduction

We know that, to date, the study of nonlinear issues and their life application is developing rapidly. Most processes in real life are represented by nonlinear specific product differential equations. No universal method has yet been developed to find an analytical solution to such equations. Only in some special cases can we find an analytical solution to nonlinear problems. For this reason, approximate solutions to such problems using numerical methods are becoming more popular. We will be able to apply approximate numerical solutions directly to life. Numerous solutions have been developed by various scientists [1-5].

2 Problem statement

Consider the problem of mutual diffusion represented by the following system of equations [6-7].

We will consider the following problem in the field $Q=\{(t,x):0<t\le T,\,0<x<L\}$

$$\begin{cases} \dfrac{\partial u}{\partial t}=\dfrac{\partial}{\partial x}\left(u^{\sigma_1}\dfrac{\partial u}{\partial x}\right)-v^{\beta_1}\left|\dfrac{\partial u}{\partial x}\right|^{p_1}, \\ \dfrac{\partial v}{\partial t}=\dfrac{\partial}{\partial x}\left(v^{\sigma_2}\dfrac{\partial v}{\partial x}\right)-u^{\beta_2}\left|\dfrac{\partial v}{\partial x}\right|^{p_2}, \end{cases} \quad (1)$$

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$$\begin{cases} u(x,0) = u_0(x), \\ v(x,0) = v_0(x). \end{cases} \quad (2)$$

Here $\sigma_1, \sigma_2, \beta_1, \beta_2, p_1, p_2$ are fixed real numbers; u^{σ_1}, v^{σ_2} are the dust and moisture permeability coefficients of the first and second media, respectively, and are functions that depend on the change in dust and humidity, respectively [3-5].

3 Differential approximation method

The differential approximation method for continuous (and sufficiently smooth) coefficient differential equations in a straight rectangular grid allows easy construction of differential schemes with first and second-order approximations. However, applying this method to more complex cases is more difficult or impossible. For example, for differential equations with a discontinuous coefficient, if the calculation area is not a right rectangle, for high-order differential equations in an uneven grid, and in other cases [4-7].

One of the following templates is selected [6,7]:

a) for the template:

$$\frac{y_n^{m+1} - y_n^m}{\tau} + c\frac{y_n^m - y_{n-1}^m}{n} = f(x_n, t_m),$$

b) for the template:

$$\frac{y_{n-1}^{m+1} - y_{n-1}^m}{\tau} + c\frac{y_n^{m+1} - y_{n-1}^{m+1}}{n} = f(x_n - 0,5h,\ t_m + 0,5\tau),$$

c) for the template:

$$\frac{y_n^{m+1} - y_n^m}{\tau} + c\frac{y_n^{m+1} - y_{n-1}^{m+1}}{h} = f(x_n - 0,5h,\ t_m + 0,5\tau),$$

d) for the template:

$$\frac{y_n^{m+1} + y_{n-1}^{m+1} - y_n^m + y_{n-1}^m}{2\tau} + c\frac{y_n^{m+1} + y_n^m - y_{n-1}^{m+1} - y_{n-1}^m}{2h} = f(x_n - 0,5h,\ t_m + 0,5\tau). \quad (3)$$

Additional conditions are approximated as follows for all cases:

$$y_n^0 = \mu_1(nh),\ n = \overline{0, N}\ ,\ h = \frac{a}{N}.$$

$$y_0^m = \mu_2(\tau m),\ m = \overline{0, M}\ ,\ \tau = \frac{T}{M}. \quad (4)$$

Let us consider a differential scheme for the following issue:

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(u^{\sigma}\frac{\partial u}{\partial x}\right)-u^{\beta}\left|\frac{\partial u}{\partial x}\right|^{p}, \tag{5}$$

$$u(x,0)=u_0(x), \tag{6}$$

$$u(0,t)=u_1(t),\ u(1,t)=u_2(t),\ 0\le t\le T. \tag{7}$$

$0\le x\le M$ with a step h to x in the cross section $\bar{\omega}_h$:

$$\bar{\omega}_h=\left\{x_i=ih,\quad h>0,\quad i=0,1,...,n,\quad hn=M\right\}$$

and for time

$$\bar{\omega}_\tau=\left\{t_j=j\tau,\quad \tau>0,\quad j=0,1,...,m,\quad \tau\ m=T\right\},T>0$$

We will build a network.

Construct the following indistinguishable scheme for the problem (5)-(7):

$$\frac{u_i^{j+1}-u_i^j}{\tau}=\frac{1}{h^2}\left(a(y_{i+1}^j)\left(u_{i+1}^{j+1}-u_i^{j+1}\right)-a\left(y_i^j\right)\left(u_i^{j+1}-u_{i-1}^{j+1}\right)\right)-Q(y)\left[\frac{u_{i+1}^j-u_i^j}{h}\right]^p.$$

Here $a(y)$ is represented by one of the following formulas [4,5]:

a) $$a(y)=\left(\frac{y_{i-1}^j+y_i^j}{2}\right)^{\sigma},$$

b) $$a(y)=\frac{\left(y_{i-1}^j\right)^{\sigma}+\left(y_i^j\right)^{\sigma}}{2}.$$

Resulting system of algebraic equations is nonlinear, and we can solve it numerically using one of the iterative methods [3-5]:

$$\frac{u_i^{j+1}-u_i^j}{\tau}=\frac{1}{h^2}\left(a(\overset{s}{y}{}_{i+1}^{j})\left(\overset{s+1}{u}{}_{i+1}^{j+1}-\overset{s+1}{u}{}_{i}^{j+1}\right)-a\left(\overset{s}{y}{}_{i}^{j}\right)\left(\overset{s+1}{u}{}_{i}^{j+1}-\overset{s+1}{u}{}_{i-1}^{j+1}\right)\right)-\left(u_i^j\right)^{\beta}Q(y).$$

$Q(y)$ is linearized by one of the following methods:

1) $$Q(y)=\left[\frac{\overset{s}{u}{}_{i+1}^{j}-\overset{s}{u}{}_{i}^{j}}{h}\right]^p$$ -Picar method [4,5],

2) $Q(y)=\dfrac{\overset{s+1}{u}{}_{i+1}^{j}-\overset{s+1}{u}{}_{i}^{j}}{h}\left[\dfrac{\overset{s}{u}{}_{i+1}^{j}-\overset{s}{u}{}_{i}^{j}}{h}\right]^{p-1}$ -special method [3,4],

3) $Q(y)=\left[\dfrac{\overset{s}{u}{}_{i+1}^{j}-\overset{s}{u}{}_{i}^{j}}{h}\right]^{p}+\beta\left(\dfrac{\overset{s}{u}{}_{i+1}^{j}-\overset{s}{u}{}_{i}^{j}}{h}\right)^{p-1}\left(\dfrac{\overset{s+1}{u}{}_{i+1}^{j}-\overset{s+1}{u}{}_{i}^{j}}{h}-\dfrac{\overset{s}{u}{}_{i+1}^{j}-\overset{s}{u}{}_{i}^{j}}{h}\right)$ - Newton's method [5,6].

Let's write the generated difference scheme as follows [4-6]:

$$A_i\overset{s+1}{y}_{i-1}-C_i\overset{s+1}{y}_{i}+B_i\overset{s+1}{y}_{i+1}=-F_i$$

Here

$$A_i=\frac{\tau}{h^2}a\left(\overset{s}{y}{}_i^j\right),\ B_i=\frac{\tau}{h^2}a\left(\overset{s}{y}{}_{i+1}^j\right)\qquad C_i=A_i+B_i+1,\ F_i=\overset{s}{y}{}_i^j-\left(u_i^j\right)^{\beta}\left[\frac{\overset{s}{u}{}_{i+1}^{j}-\overset{s}{u}{}_{i}^{j}}{h}\right]^{p}\tau.$$

According to the Picard method,

$$A_i=\frac{\tau}{h^2}a\left(\overset{s}{y}{}_i^j\right),\ B_i=\frac{\tau}{h^2}a\left(\overset{s}{y}{}_{i+1}^j\right)+\frac{\overset{s+1}{y}{}_{i+1}^{j}}{h}\left[\frac{\overset{s}{y}{}_{i+1}^{j}-\overset{s}{y}{}_{i}^{j}}{h}\right]^{p-1}\left(v_i^j\right)^{\beta}\tau,$$

$$C_i=A_i+B_i+1,\ F_i=\overset{s}{y}{}_i^j.$$

According to a special method,

$$A_i=\frac{\tau}{h^2}a\left(\overset{s}{y}{}_i^j\right),$$

$$B_i=\frac{\tau}{h^2}a\left(\overset{s}{y}{}_{i+1}^j\right)+p\left(u_i^j\right)^{\beta}\left(\frac{\overset{s}{y}{}_{i+1}^{j}-\overset{s}{y}{}_{i}^{j}}{h}\right)^{p-1}\frac{\overset{s+1}{y}{}_{i+1}^{j}}{h}\tau.$$

According to Newton's method,

$$C_i=A_i+B_i+1,$$

$$F_i=\overset{s}{y}{}_i^j-\left(u_i^j\right)^{\beta}\left[\frac{\overset{s}{u}{}_{i+1}^{j}-\overset{s}{u}{}_{i}^{j}}{h}\right]^{p}\tau+p\left(u_i^j\right)^{\beta}\left(\frac{\overset{s}{y}{}_{i+1}^{j}-\overset{s}{y}{}_{i}^{j}}{h}\right)^{p}\tau.$$

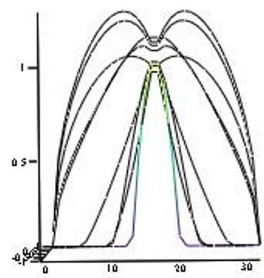
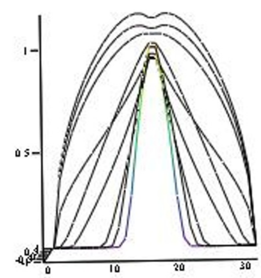
Let us now consider the algorithm for solving the problem (1) - (2) numerically.

$$\begin{cases} \dfrac{u_i^{j+1}-u_i^j}{\tau} = \dfrac{1}{h^2}\left(a(y_{i+1}^j)\left(u_{i+1}^{j+1}-u_i^{j+1}\right)-a\left(y_i^j\right)\left(u_i^{j+1}-u_{i-1}^{j+1}\right)\right)-Q(y)\left(v_i^j\right)^{\beta_1}, \\ \dfrac{v_i^{j+1}-v_i^j}{\tau} = \dfrac{1}{h^2}\left(a(y_{i+1}^j)\left(v_{i+1}^{j+1}-v_i^{j+1}\right)-a\left(y_i^j\right)\left(v_i^{j+1}-v_{i-1}^{j+1}\right)\right)-Q(y)\left(u_i^j\right)^{\beta_2}. \end{cases}$$

The resulting system of algebraic equations is nonlinear, and we can solve it numerically using one of the iterative methods and the sweep method [8,9].

$$\begin{cases} \dfrac{\overset{s+1}{u_i^{j+1}}-\overset{s}{u_i^j}}{\tau} = \dfrac{1}{h^2}\left(a(\overset{s}{y_{i+1}^j})\left(\overset{s+1}{u_{i+1}^{j+1}}-\overset{s+1}{u_i^{j+1}}\right)-a\left(\overset{s}{y_i^j}\right)\left(\overset{s+1}{u_i^{j+1}}-\overset{s+1}{u_{i-1}^{j+1}}\right)\right)-Q(y)\left(v_i^j\right)^{\beta_1} \\ \dfrac{\overset{s+1}{v_i^{j+1}}-\overset{s}{v_i^j}}{\tau} = \dfrac{1}{h^2}\left(a(\overset{s}{y_{i+1}^j})\left(\overset{s+1}{v_{i+1}^{j+1}}-\overset{s+1}{v_i^{j+1}}\right)-a\left(\overset{s}{y_i^j}\right)\left(\overset{s+1}{v_i^{j+1}}-\overset{s+1}{v_{i-1}^{j+1}}\right)\right)-Q(y)\left(u_i^j\right)^{\beta_2} \end{cases}$$

Below are the results of this numerical experiment.

Parameter values	Results of the computational experiment
$\sigma_1 = 1.1, \sigma_2 = 1.3$ $\beta_1 = 1,\ k_1 = 7$ $\beta_2 = 1,\ k_2 = 9$ $eps = 10^{-3}$	
$\sigma_1 = 1.3, \sigma_2 = 1.5$ $\beta_1 = 1,\ k_1 = 3$ $\beta_2 = 1,\ k_2 = 2$ $eps = 10^{-3}$	

A solution was obtained for another view of the mutual diffusion problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(v^{\sigma_1}\dfrac{\partial u}{\partial x}\right) \pm \gamma(t,x)\dfrac{\partial u}{\partial x}, \\ \dfrac{\partial v}{\partial t} = \dfrac{\partial}{\partial x}\left(v^{\sigma_2}\dfrac{\partial v}{\partial x}\right) \pm \gamma(t,x)\dfrac{\partial v}{\partial x}, \end{cases}$$

$$u|_{t=0} = u_0(x) \geq 0; \qquad v|_{t=0} = v_0(x) \geq 0,$$

$$u|_{x=x_1} = \psi_1(t) \geq 0; \qquad v|_{x=x_1} = \varphi_1(t) \geq 0,$$

$$u|_{x=x_2} = \psi_2(t) \geq 0; \qquad v|_{x=x_2} = \varphi_2(t) \geq 0.$$

Here $u(t,x) -$ is dust;

$v(t,x) -$ is humidity;

$\sigma_1 \geq 0; \quad \sigma_2 \geq 0$ – are real numbers;

$u^{\sigma_1}\frac{\partial u}{\partial x}$ – is dust transfer coefficient;

$v^{\sigma_2}\frac{\partial v}{\partial x}$ – is humidity transfer coefficient;

$\gamma(t,x)$ – is ambient speed.

The system of given equations $\gamma(t,x) = \gamma(t)$ with component $\gamma(t,x)$ represents a nonlinear, convective diffusion moisture-dust migration process moving at speed.

The numerical solution of this problem was also solved as a problem represented by the system of equations (1) - (2), and the results were obtained [8, 9].

Some results obtained for the dust and moisture migration process in a two-component environment are presented graphically.

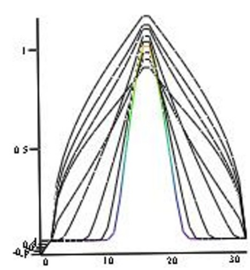
Parameter values — Results of the computational experiment

$\sigma_1 = 1.1, \sigma_2 = 1.3$

$k_1 = 0.1$

$k_2 = 0.25$

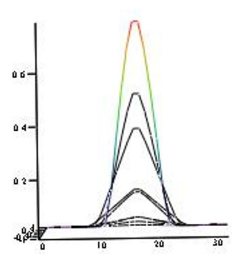
$eps = 10^{-3}$



$k_1 = 5$

$k_2 = 7$

$eps = 10^{-3}$



It can be expected that further theoretical and experimental studies of excitable systems with cross-diffusion will significantly contribute to the study of self-organization phenomena in all nonlinear systems, from micro- and astrophysical systems to social systems.

4 Conclusion

Nonlinear processes of multicomponent competing systems are numerically investigated, and the analysis of the results based on the obtained estimates of solutions is carried out, which shows the high performance of algorithms and software packages in finding new effects for solving a system of parabolic equations. The developed numerical schemes, algorithms, and software packages make it possible to carry out computer simulations to study reaction-diffusion processes based on the qualitative properties of a nonlinear mathematical model and determine the appearance of a dissipative structure.

We know that the system of equations of the parabolic type can be solved analytically only in some special cases. Therefore, numerical solutions were used, and solutions close to the exact solution were obtained. As a result of visualizing the results, we see the movement of the process and have the opportunity to express a clear opinion about it. It is no exaggeration to say that this is a step towards finding solutions to many practical problems.

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