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The method of averaging over a movable liquid control volume for some problems of fluid mechanics

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Abstract. The paper considers the method of averaging over a movable liquid control volume for some problems of fluid mechanics. The authors study the process of obtaining a solution for the problem of fluid flow with different configurations along the cross section of a pipe with a filled porous medium, as well as flow in an open-flow channel. Obtaining an approximate analytical solution based on a movable control volume is described. The well-known control volume method used in numerical analysis is used considering its displacement. The method of displacement makes it possible to obtain an analytical representation of the solution of the problem under consideration. At the same time, obtaining an analytical solution method is achieved by averaging the equation describing the flows over the control volume. Based on the obtained solution in the limit, we obtain solutions to the problem without considering porous media, and with different pipe cross-sections (flat, round, ellipsoidal and rectangular). With certain configurations of the pipe section, an exact solution is obtained.

1 Introduction

Many problem statements in which the flow in pipes filled with porous inclusions is studied are typical for modern technological processes and devices in mining and machine-building enterprises. Depending on the technologies and devices used, it can be noted that the sections of these pipes also have a different shape. As a rule, these are round, rectangular, elliptical and other cross-sectional shapes. To determine the patterns of flow, it is important to know the hydrodynamic characteristics of the flow in such pipes. For practice, visualization and compactness of solving problems of studying patterns are important, since a practical engineer works with this data. Therefore, it is convenient for a specialist to obtain solutions in a compact form. This allows him to draw conclusions about the nature of the flow in pipes and channels in real time without delay associated with lengthy data processing.

It is known that many problems on the flow of liquid and gas belong to the class of research problems, which are reduced to solving boundary value problems of differential equations. Based on the requirements of our formulation of the problem, it is important to

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use approximate solution methods that have less accuracy, but allow solving the problem quickly. Currently, various approximate analytical methods are used that allow solving differential equations. For example, the use of an artificial neural network. In this connection, we note the works [1, 2]. In these works, the authors propose to use an artificial neural network based on models for solving linear ordinary differential equations. In this case, solutions are considered for linear ordinary differential equations of the first and second order with a constant coefficient under initial conditions. In contrast to the standard approach, in [3-7] for solving differential equations, the authors proposed and implemented some non-standard difference schemes.

We can note the method of moving volumes. This method turned out to be a fairly convenient way to obtain an approximate analytical solution of differential equations. The method is applied on the basis of a movable node. In [8], an approximate analytical expression for the difference solution of a differential problem was obtained. The movable node method was also applied in [9]. In this work, on the basis of the method of movable nodes, the determination and construction of control volumes is carried out. Combinations of the movable node method with the ideas of Richardson's extrapolation are aimed at improving the accuracy of calculations. This is substantiated in [10]. Also, a number of authors solve the problems of monotonicity of the difference scheme using a movable node. One approach to the solution is presented in [11]. The number of applied solutions that are associated with the use of this method in technological processes and devices of production systems is growing. Some applied problems, for which the movable node method was applied, are presented in [12]. Based on the choice of the velocity profile on the edge of the control volume, qualitative schemes were obtained in [13].

2 Materials and methods

In this work, we will consider the process as the process of the flow of a viscous Newtonian fluid through pipes filled with a porous medium. Since we believe that the flow is stationary and one-dimensional, then the Rakhmatulin equation [14-16] is applicable. Based on this, we get the following equation in the Cartesian coordinate system

$$\frac{\partial}{\partial x}\left(f\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(f\frac{\partial u}{\partial y}\right)=-f\frac{\Delta p}{\mu l}+Ku \tag{1}$$

Here $u(x,y)$ is the flow velocity, the Oz axis is directed along the pipe axis, $\Delta P/l$ is the pressure drop, μ is the viscosity of the liquid, f is the porosity of the medium, K is the interaction coefficient obtained on the basis of the Kozeni–Karman ratio $\left(K=\frac{\alpha}{d^2}\left(\frac{1-f}{f}\right)^2\right)$, d is the characteristic size of the porous medium (we consider the pressure drop and the viscosity of the liquid constant). We consider the pipe boundary satisfying the equation

$$\frac{x^n}{a^n}+\frac{y^n}{b^n}=1 \tag{2}$$

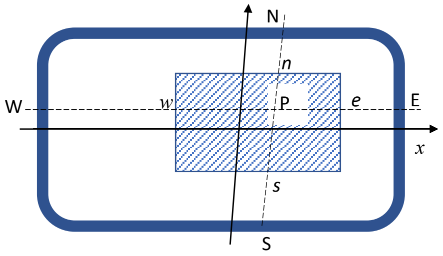

Fig. 1. Pipe section.

The pipe section is shown in figure 1. In the case of $n = 2$and $a = b$ (2) represents the cross section of a circular pipe with radius a; for $n = 2$, $a \neq b$, we obtain the cross section of an ellipsoidal pipe with semi-axes a and b. With sufficiently large even n, the pipe section approaches a rectangular one.

Thus, to find a solution to equation (1) with the conditions of adhesion at the pipe boundaries by the moved volume method.

This method is also used for fluid flow problems in open channels.

3 Results and discussion

3.1 Solving the problem of flow through pipes by the method of a movable control volume

Let $P(x, y)$ be an arbitrary point inside the pipe section (figure 1). We make parallel lines through this point along the coordinate axes to the intersection with the pipe boundary. Denote these points by E, W, N, S. The coordinates of these boundary points: $E(x_E, y_E), W(x_W, y_W)$, $N(x_N, y_N)$ and $S(x_S, y_S)$. Select a rectangular control volume as follows. Let's choose the midpoint of the points $P$and E ($e = 0.5(E + P)$), W and P ($w = 0.5(W + P)$), similarly $n = 0.5(N + P)$ and $s = 0.5(S + P)$. Let's denote the coordinates of the points $e(x_e, y_e)$, $w(x_w, y_w)$, $n(x_n, y_n)$ and $s(x_s, y_s)$. By making parallel lines through these points along the coordinate axes, we get a rectangle (the shaded area in figure 1). When P changes its position, the control volume also changes. We integrate equation (1) over the control volume.

$$\iint_D \left[\frac{\partial}{\partial x}\left(f\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(f\frac{\partial u}{\partial y}\right)\right] dxdy = \iint_D [fA + Ku] dxdy. \tag{3}$$

Here $D: (x_w \leq x \leq x_e; y_s \leq y \leq y_n)$, $A = -\Delta P/(\mu l)$. The first term in (3) is transformed as follows:

$$\iint_D \frac{\partial}{\partial x}\left(f\frac{\partial u}{\partial x}\right) dxdy = \int_{y_s}^{y_n} dy \int_{x_w}^{x_e} \frac{\partial}{\partial x}\left(f\frac{\partial u}{\partial x}\right) dx = \frac{y_N - y_S}{2}\left[f\frac{\partial u}{\partial x}\bigg|_{x_e} - f\frac{\partial u}{\partial x}\bigg|_{x_w}\right]$$
$$\approx \frac{y_N - y_S}{2}\left(f_{x_e}\frac{u_E - u}{x_E - x} - f_{x_w}\frac{u - u_W}{x - x_W}\right).$$

Similarly, the second term of the left part is approximated. The integral in the right part (3) is replaced by the expression

$$\iint_D [fA + Ku] dxdy \approx (fA + Ku)\frac{x_E - x_W}{2}\cdot\frac{y_N - y_S}{2}.$$

Thus, (2) is replaced by the following algebraic equation

$$\frac{2}{x_E - x_W}\left[f(e)\frac{u_E - u}{x_E - x} - f(w)\frac{u - u_W}{x - x_W}\right] + \frac{2}{y_N - y_S}\left[f(n)\frac{u_N - u}{y_N - y} - f(s)\frac{u - u_W}{y - y_S}\right]$$
$$= (fA + Ku).$$

Using the no-slip boundary conditions, we find the flow velocity

$$u = \frac{2G\cdot H}{2[f(e)(x - x_W) + f(w)(x_E - x)]G + [f(n)(y - y_S) + f(s)(y_N - y)]H + K\cdot G\cdot H}\cdot\frac{f(P)\cdot A}{2} \tag{4}$$

Where the designation is entered

$$H = (x_E - x_W)(x_E - x)(x - x_W), G = (y_N - y_S)(y_N - y)(y - y_S).$$

(4) represents the distribution of the flow velocity for pipes of different cross-sections filled with a porous medium.

Due to the fact that the points are on the boundary (2), we get,

$$y_N = b\sqrt[n]{1-\frac{x^n}{a^n}}, y_S = -b\sqrt[n]{1-\frac{x^n}{a^n}}, x_E = a\sqrt[n]{1-\frac{y^n}{b^n}}, x_W = a\sqrt[n]{1-\frac{y^n}{b^n}}. \tag{5}$$

If we assume in (4), we get the solution of the problem in the absence of a porous layer, we get,

$$u = \frac{a^2b^2\left[\sqrt[n]{\left(1-\frac{y^n}{b^n}\right)^2}-\frac{x^2}{a^2}\right]\cdot\left[\sqrt[n]{\left(1-\frac{x^n}{a^n}\right)^2}-\frac{y^2}{b^2}\right]}{a^2\left[\sqrt[n]{\left(1-\frac{y^n}{b^n}\right)^2}-\frac{x^2}{a^2}\right]+b^2\left[\sqrt[n]{\left(1-\frac{x^n}{a^n}\right)^2}-\frac{y^2}{b^2}\right]}\cdot\frac{A}{2} \tag{6}$$

Let us consider special cases. Let's put in (6) $n = 2$ and $a = b$, i.e. consider the flow in a circular tube with radius a. Then we have

$$u = \frac{a^2}{4}\left(1-\frac{r^2}{a^2}\right)\cdot\frac{A}{2} \tag{7}$$

Here $r^2 = x^2 + y^2$. Thus, we obtain Poiseuille's law for a round pipe. If $n = 2$, and $a \neq b$ then we have a flow in an ellipsoidal tube. In this case, from (6) follows

$$u = \frac{a^2b^2}{2(a^2+b^2)}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\cdot\frac{A}{2} \tag{8}$$

The law of the distribution of the flow velocity according to the formula (9) coincides with the exact solution [18].

To obtain an approximate analytical solution for a pipe with a rectangular cross-section, we proceed to the limit in (6) at $n \to \infty$. After a simple operation, we get

$$u = \frac{a^2b^2\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)}{a^2\left(1-\frac{x^2}{a^2}\right)+b^2\left(1-\frac{y^2}{b^2}\right)}\cdot\frac{A}{2} \tag{9}$$

However, formula (9) does not coincide with the exact solution [18-20], but it comes close to it. To get a solution for a flat pipe, we go to the limit in (9) at $a \to \infty$, then we have

$$u = b^2\left(1-\frac{y^2}{b^2}\right)\cdot\frac{A}{2} \tag{10}$$

Solution (10) accurately reflects the solution of the problem for a flat pipe with height [18].

Let's compare the exact solution for a rectangular pipe, which has the form [18]

$$u = A\frac{16a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\left[1-\frac{ch\left(\frac{2n+1\pi}{2\ \ a}y\right)}{ch\left(\frac{2n+1\pi b}{2\ \ a}\right)}\right]cos\left(\frac{2n+1}{2}\frac{\pi}{a}x\right) \tag{11}$$

with an approximate solution (9) ($b$the height of the rectangle, the parallel axis Oy, the base of the parallel axis Ox and the width is $2a$ (the axis passes through the center of the rectangle and is directed downstream). Figure 2 shows the exact and approximate solution at the in section $x = 1$ ($A = 1, b = 1, a = 2.5$).

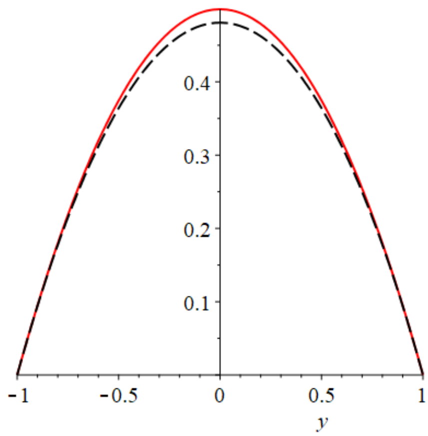

Fig. 2. Comparison of the solution.

Thus, formula (6) describes the velocity distribution well for pipes of type (2) with different cross-sections.

3.2 Solving the problem of open channel flow by the method of a movable control volume

Let us consider the currents in a long channel. There is no pressure drop along the length of the flow, the transverse pressure drop is static, the same in all sections.

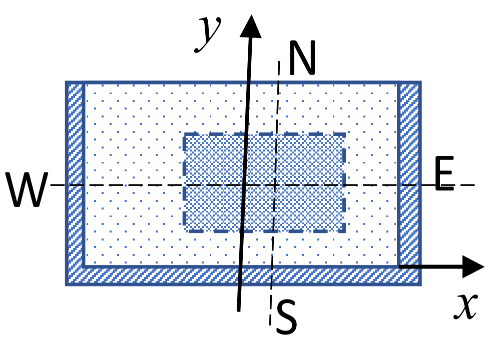

Fig. 3. Rectangular channel.

Let the angle of inclination to the horizon is α. Then from the Navier-Stokes equation [18] we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\frac{\rho g \sin\alpha}{\mu} \tag{12}$$

Let us consider the section of the channel in the form of a rectangle. The height of the channel h, and the width $2b$. Boundary conditions are no-slip. The condition on the free boundary is

$$\frac{\partial u}{\partial y} = 0. \tag{13}$$

Integrating (13) over the control volume (figure 3) we get

$$\frac{2}{x_E - x_W}\left[\frac{u_E - u}{x_E - x} - \frac{u - u_W}{x - x_W}\right] + \frac{2}{y_N - y_S}\left[\frac{u_N - u}{y_N - y} - \frac{u - u_W}{y - y_S}\right] = A. \tag{14}$$

Here $A = -\frac{\rho g \sin\alpha}{\mu}$.

From the no-slip condition it follows that $u_E = u_W = u_S = 0$. For the definition $u_N$we proceed as follows. In (14), let's go to the limit $y \to y_N$then, we get

$$\frac{u_N}{b}\left[\frac{1}{b-x} + \frac{1}{x+b}\right] + \frac{2}{h}\left[\frac{\partial u(x, y_N)}{\partial y} - \frac{u_N}{h}\right] = A.$$

Hence, using (14) we define

$$u_N = -A\frac{h^2(b-x)(x+b)}{2h^2 + 2(b^2 - x^2)} \tag{15}$$

Using the no-slip conditions and (16) from equation (15) we obtain

$$u = -A\frac{(b^2 - x^2)y}{y(h-y) + b^2 - x^2}\left[\frac{h-y}{2} + \frac{h(b^2 - x^2)}{2(h^2 + b^2 - x^2)}\right]. \tag{16}$$

Let us now consider the velocity distribution along the trapezoidal channel. The characteristics of the channel and the coordinate system are shown in figure 4. Due to the symmetry of the flow, relative to the axis, consider the right half of the channel. In this case, using the same technique, we obtain a solution to the flow problem as follows

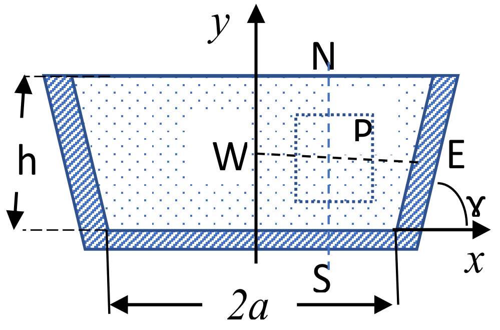

Fig. 4. Trapezoidal channel.

$$u = \frac{x_E - x}{x(x_E - x) + (b-y)(y-y_S)}\left[-\frac{A}{2}x(h-y)(y-y_S) + \frac{(h-y)(y-y_S)u_W}{x_E} + xu_N\right]. \tag{17}$$

here,

$$u_N = \frac{A}{2}\left[x_E - x + \frac{(h-y)(y-y_S)x_E^2}{V}\right]\frac{V(b-y_S)(y-y_S)}{x(y-y_S)^3 x_E^2 - [x(x_E - x) + (y-y_S)^2](y-y_S)V},$$

$$u_W = \frac{A}{2}(x_E - x) + \frac{x(x_E - x) + (y-y_S)^2}{x(y-y_S)^2}u_N,\ V = x_E^2 + (h-y)(y-y_S),$$

$$y_S = \begin{cases} 0 if x_E \le a, \\ (x_E - a)tg\gamma if x_E > a, a < x_E \le a + h/tg\gamma \end{cases}$$

4 Conclusion

The movable control volume method is a good way to obtain analytical solutions to a number of problems in mathematical physics. It should be noted that using this method, the solution of the problem of flow in pipes is highly versatile and allows one to obtain a solution to the problem for pipes with different configurations and different porous fillings. This simple type of decision enables specialists to analyze the results obtained and allows them to make informed decisions during the study.

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