Proper normal waves in a two-layer tube taking into account the rheological properties of materials

Ismoil Safarov[1], *Mukhsin* Teshaev[2], *Tulkin* Ruziev[3], *Matlab* Ishmamatov[4*], and *Nurillo* Kulmuratov[4]

[1]Depatrment of Mathematics, Tashkent Institute of Chemical Technology, 32, Navoi str., Tashkent city, Uzbekistan
[2]Bukhara branch of Institute of Mathematics named after V.I. Romanovski, 11, M.Iqbol str., Bukhara city, Uzbekistan
[3]Department of Natural and Exact Sciences, Bukhara State Pedagogical Institute, 16, K. Murtazayev str., Bukhara city, Uzbekistan
[4]Depatrment of Mathematics, Navoi Mining and Technology University, 27, Galaba str., Navoi sity, Uzbekistan

Abstract. The article considers the problem of propagation of proper normal waves in a two-layer pipe, paying to attention the rheological properties of the material. The main purpose of the work is to develop a methodology, an algorithm for investigation the problems of wave propagation in extended two-layer dissipatively inhomogeneous cylindrical structures. The main integro-differential equations in the article are obtained on the basis of methods of the theory of visco-elasticity. In the course of the study, the method of separation of variables, the freezing method, the Muller method and the Godunov orthogonal run method were used. In this paper, for structurally inhomogeneous extended cylindrical mechanical systems, the dependences of several modes (three and four) of the complex phase velocity (real and imaginary parts) on the wave number are comparatively estimated; the results of calculations are compared with experimental and theoretical data of other researchers, and the use of asymptotic and numerical methods for solving dispersion equations with complex output parameters is justified.

1 Introduction

The problems of wave propagation in continuous multilayer systems attract the attention of numerous researchers in our country and abroad. This is due to the fact that in many fields of science and technology, it is increasingly necessary to face the need to calculate stress and strain fields that occur in layered bodies with different rheological properties when exposed to various kinds of dynamic loads [1,2]. Dynamic problems of dissipative (viscoelastic) dynamical systems are solved by methods of mathematical physics. The complexity of their solution is explained by many factors, such as rheological properties of media (anisotropy,

* Corresponding author: matlab1962@mail.ru

viscosity, creep, plasticity, heterogeneity, etc.), which generates a wide variety of models for describing real phenomena and does not allow creating a single mathematical model of the process in a mechanical system. Despite the large number of mathematical models of a mechanical system, mathematical methods for solving problems have been developed mainly for such systems as acoustic, whose elastic motions are described by linear differential equations [3,4]. In [5,6,7], an attempt was made to determine and optimize the dissipative characteristics, as well as the stress-strain states of mechanical systems. In the works mentioned above, two modes of operation of the system are considered - natural and forced oscillations. Natural oscillations mean movements in which all points of the system oscillate with the same frequencies and damping indicators (but with different amplitudes). In the mathematical formulation of the problem of excitation and propagation of waves in an extended elastic plate body, certain problems with boundary conditions at infinity appear. Sommerfeld radiation conditions do not always provide a unique solution to the problem. Also, when taking into account Rayleigh waves that propagate in space, we again have to estimate free waves at infinity. Changes in the physical properties of the material add additional difficulties. Similar difficulties arise in the propagation of normal waves in waveguides with variable cross-sections. When the cross-section of the plate is constant, studies on wave propagation were carried out in [8,9]. In a number of works, theoretical and experimental studies on the propagation of the phase velocity of natural waves, as well as the group velocity, depending on the physical properties of the material, were carried out on the basis of numerical methods [10]. Based on this, the equation of motion for the propagation of waves in the plate and shell are obtained to simplify the tasks.

In some papers [11, 12], the characteristics of normal modes of natural waves in extended elastic plate bodies were investigated. In [13], the properties of normal modes of natural waves in plate deformable bodies have a number of features that have no analogue for modes in acoustic and electromagnetic waveguides.

2 Methods

2.1 Problem statement and solution methods

Consider a two-layer viscoelastic pipe (Figure 1) with internal

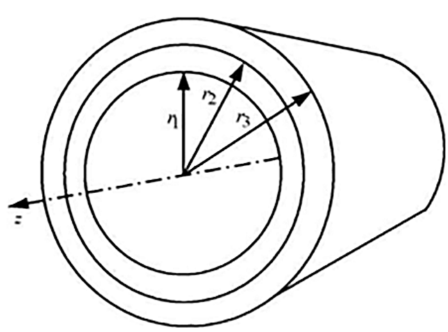

Fig. 1. Calculation scheme.

The radiusr_1, the outer radiusr_3 , and the radius of the boundary between layers r_2 . In a cylindrical coordinate system, the axis is directed along the central axis of the cylinder. The cross-sectional space of the pipe is given in the form $r_1 \leq r \leq r_2$, which is denoted by I, the second, outer cylinder is given with radii $r_2 \leq r \leq r_3$, and is denoted by II .

The basic equations of motion of a two-layer pipe are given by three groups of linear relations of the theory of viscoelasticity [14-15]

$$\frac{\partial \sigma_{ij}^{(n)}}{\partial x_j} + \rho_k F_i^{(n)} = \rho \frac{\partial^2 u_{in}}{\partial t^2}, \tag{1}$$

$$\varepsilon_{ij}^{(n)} = \frac{1}{2}\left(\frac{\partial u_{in}}{\partial x_j} + \frac{\partial u_{jn}}{\partial x_i}\right),$$

$$\sigma_{ij}^{(n)} = \tilde{\lambda}_n \varepsilon_{kk}^{(n)} \delta_{ij} + 2\tilde{\mu}_n \varepsilon_{ij}^{(n)}$$

Here $\sigma_{ij}^{(n)}, \varepsilon_{ij}^{(n)}$ are the stresses and strains (n=1,2), $\tilde{\lambda}_n, \tilde{\mu}_n$ are operator modulus of elasticity:

$$\tilde{\lambda}_n[\phi(t)] = \frac{v_n \tilde{E}_n}{(1+v_n)(1-2v_n)}[\phi(t)] = \frac{v_n E_{0n}}{(1+v_n)(1-2v_n)}\left[\phi(t) - \int_0^t R_{En}(t-\tau)\phi(\tau)d\tau\right];$$

$$\tilde{\mu}_n \phi(t) = \frac{v_n \tilde{E}_n}{2(1+v_n)}[\phi(t)] = \frac{v_n E_{0n}}{2(1+v_n)}\left[\phi(t) - \int_0^t R_{En}(t-\tau)\phi(\tau)d\tau\right], \tag{2}$$

v_n is the Poisson's ratio, $R_{En}(t-\tau)$ is the relaxation cernel, E_{0n} is the instantaneous modulus of elasticity [14], $\phi(t)$ is the arbitrary function of time.

Then we apply the freezing method

$$\bar{\lambda}_n = \lambda_{0n}\left[1 - \Gamma_{\lambda n}^C(\omega_R) - i\Gamma_{\lambda n}^S(\omega_R)\right] = \lambda_{0n}\Gamma_{\lambda n};$$

$$\bar{\mu}_n = \mu_{0n}\left[1 - \Gamma_{\mu n}^C(\omega_R) - i\Gamma_{\mu n}^S(\omega_R)\right] = \mu_{0n}\Gamma_{\mu n},$$

$$\lambda_{0n} = \frac{v_n E_{0n}}{(1+v_n)(1-2v_n)}, \mu_{0n} = \frac{v_n E_{0n}}{2(1+v_n)}, \tag{3}$$

instantaneous Lame coefficients. The calculations used a three-parameter Koltunov-Rzhanitsyn relaxation kernel.

Boundary conditions:

$$\sigma_{rr}|r = r_1 = \sigma_{r\theta}|r = r_1 = \sigma_{rz}|r = r_1 = 0,$$

$$\sigma_{rr}|r = r_3 = \sigma_{r\theta}|r = r_3 = \sigma_{rz}|r = r_3 = 0, \tag{4}$$

$$\sigma_{rr}^{(1)}|r = r_2 = \sigma_{rr}^{(2)}|r = r_2, \sigma_{r\theta}^{(1)}|r = r_2 = \sigma_{r\theta}^{(2)}|r = r_2, \sigma_{rz}^{(1)}|r = r_2 = \sigma_{rz}^{(2)}|r = r_2,$$

$$u_{r1}|r = r_2 = u_{r2}|r = r_2, u_{\theta 1}|r = r_2 = u_{\theta 2}|r = r_2, u_{z1}|r = r_2 = u_{z2}|r = r_2.$$

For pipe displacements, the Green–Lame decomposition is valid [2]:

$$\vec{u}_n = grad\varphi_n + rot\vec{\psi}_n, div\vec{\psi}_n = 0,$$

where $\vec{u}_n(u_{rn}, u_{\theta n}, u_{zn})$ is the displacement vector of the medium, φ_n and $\vec{\psi}(\psi_{rn}, \psi_{\theta n}, \psi_{zn})$ - are the potentials of longitudinal and transverse waves. Then the equations of motion of a two-layer pipe (1), taking into account (2)-(4) with respect to potential functions, in a cylindrical coordinate system take the following form:

$$\Gamma_{\lambda\mu n}^{\bullet}\nabla^2\varphi_n - \frac{1}{c_{pn}^2}\frac{\partial^2\varphi_n}{\partial t^2} = 0,$$

$$\Gamma_{\mu n}^{\bullet}\nabla^2\psi_{zn} - \frac{1}{c_{sn}^2}\frac{\partial^2\psi_{zn}}{\partial t^2} = 0, \tag{5}$$

$$\Gamma_{\mu n}(\nabla^2\psi_{\theta n} - \frac{\psi_{\theta n}}{r^2} + \frac{2}{r^2}\frac{\partial\psi_{rn}}{\partial\theta}) - \frac{1}{c_{s\,0n}^2}\frac{\partial^2\psi_{\theta n}}{\partial t^2} = 0,$$

$$\Gamma_{\mu n}(\nabla^2\psi_{rn} - \frac{\psi_{rn}}{r^2} - \frac{2}{r^2}\frac{\partial\psi_{\theta n}}{\partial\theta}) - \frac{1}{c_{s\,0n}^2}\frac{\partial^2\psi_{rn}}{\partial t^2} = 0.$$

where $\Gamma_{\lambda\mu\kappa}^{\bullet} = 1 - \Gamma_{\lambda\mu\kappa}^C(\omega_R) - i\Gamma_{\lambda\mu\kappa}^S(\omega_R), \Gamma_{\mu\kappa}^{\bullet} = 1 - \Gamma_{\mu\kappa}^C(\omega_R) - i\Gamma_{\mu\kappa}^S(\omega_R)$; $\Gamma_{\lambda\mu\kappa}^C(\omega_R)$, $\Gamma_{\lambda\mu\kappa}^S(\omega_R)$, $\Gamma_{\mu\kappa}^C(\omega_R)$, $\Gamma_{\mu\kappa}^S(\omega_R)$ accordingly, the sine and cosine Fourier images of relaxation kores are determined similarly (5); $c_{pn}^2 = (\lambda_{n0} + 2\mu_{n0})/\rho_n, c_{sn}^2 = \mu_{n0}/\rho_n$ - are the longitudinal and transverse wave propagation velocities.

The solution of the system (5) is sought in the form

$$\varphi_n(r,\theta,z,t)=\sum_{n=0}^{\infty}\phi_n(\alpha_n r)\begin{Bmatrix}\cos k\theta\\ -\sin k\theta\end{Bmatrix}e^{\pm i\gamma_p z}e^{-i\omega t};$$
$$\psi_{rn}(r,\theta,z,t)=\sum_{n=0}^{\infty}\Psi_{nr}(\beta_n r)\begin{Bmatrix}\sin k\theta\\ -\cos k\theta\end{Bmatrix}e^{\pm i\gamma_p z}e^{-i\omega t};$$
$$\psi_{\theta n}(r,\theta,z,t)=\sum_{n=0}^{\infty}\Psi_{n\theta}(\beta_n r)\begin{Bmatrix}\cos k\theta\\ -\sin k\theta\end{Bmatrix}e^{\pm i\gamma_p z}e^{-i\omega t};$$
$$\psi_{zn}(r,\theta,z,t)=\sum_{n=0}^{\infty}\Psi_{nz}(\beta_n r)\begin{Bmatrix}\sin k\theta\\ \cos k\theta\end{Bmatrix}e^{\pm i\gamma_p z}e^{-i\omega t}; \quad (6)$$

where k is an integer; $\gamma_p=\omega/c$ - wave number; $\omega=\omega_R+i\omega_I$ -complex natural frequency; $r=\frac{r_1}{r_1}, z=\frac{z_1}{r_1}$. Substituting (6) into the equation of motion (5), we obtain the following expression $\phi_n, \psi_{zn}, \phi_{\theta n}, \phi_{rn}$:

$$\frac{d^2\phi_n}{dr^2}+\frac{1}{r}\frac{d\phi_n}{dr}+\left(\alpha_n^2-\frac{k^2}{r^2}\right)\phi_n=0, \quad (7)$$
$$\frac{d^2\Psi_{zn}}{dr^2}+\frac{1}{r}\frac{d\Psi_{zn}}{dr}+\left(\beta_n^2-\frac{k^2}{r^2}\right)\Psi_{zn}=0,$$
$$\frac{d^2\Psi_{\theta n}}{dr^2}+\frac{1}{r}\frac{d\Psi_{\theta n}}{dr}+\frac{1}{r^2}\left(-k^2\Psi_{\theta k}-2k\Psi_{\theta k}-\Psi_{\theta k}\right)+\beta_n^2\Psi_{\theta k}=0, \quad (8)$$
$$\frac{d^2\Psi_{rn}}{dr^2}+\frac{1}{r}\frac{d\Psi_{rn}}{dr}+\frac{1}{r^2}\left(-k^2\Psi_{rn}+2k\Psi_{\theta n}-\Psi_{rn}\right)+\beta_n^2\Psi_{rn}=0, \quad (9)$$

where

$$\alpha_n^2=\frac{\Omega_n^2}{\gamma_{1n}^2}-\gamma_p^2, \Omega_n^2=\frac{\omega r_n}{c_{s0k}}; \quad \gamma_{1n}^2=\frac{2(1-\nu_{0n})\Gamma_{\lambda\mu n}}{1-2\nu_{0n}},$$
$$\beta_n^2=\Omega_k^2/\gamma_{2n}^2-\gamma_p^2, \gamma_{2n}^2=\Gamma_{\mu n}.$$

Subtracting and adding the last two equations (8) and (9), we get:

$$\frac{d^2}{dr^2}(\Psi_{rn}-\Psi_{\theta n})+\frac{1}{r}\frac{d}{dr}(\Psi_{rn}-\Psi_{\theta n})+\left(\beta_n^2-\frac{(k+1)^2}{r^2}\right)(\Psi_{rn}-\Psi_{\theta n})=0,$$
$$\frac{d^2}{dr^2}(\Psi_{rn}+\Psi_{\theta n})+\frac{1}{r}\frac{d}{dr}(\Psi_{rn}+\Psi_{\theta n})+\left(\beta_n^2-\frac{(k+1)^2}{r^2}\right)(\Psi_{rn}+\Psi_{\theta n})=0. \quad (10)$$

Solutions of equations (7)-(9) have the form:

$$\phi_1(r)=A_{1m}Z_m(\alpha_1 r)+A_{2m}W_m(\alpha_1 r);$$
$$\Psi_{z1}(r)=A_{3m}Z_m(\beta_1 r)+A_{4m}W_m(\beta_1 r);$$
$$\Psi_{r1}(r)-\Psi_{\theta 1}(r)=2A_{5m}Z_{m+1}(\beta_1 r)+2A_{6m}W_{m+1}(\beta_1 r);$$
$$\Psi_{r1}(r)+\Psi_{\theta 1}(r)=2A_{13m}Z_{n-1}(\beta_1 r)+2A_{14m}W_{m-1}(\beta_1 r);$$
$$\phi_2(r)=A_{7m}Z_m(\alpha_1 r)+A_{8m}W_m(\alpha_1 r);$$
$$\Psi_{z2}(r)=A_{9m}Z_m(\beta_1 r)+A_{10m}W_m(\beta_1 r);$$
$$\Psi_{r2}(r)-\Psi_{\theta 2}(r)=2A_{11m}Z_{m+1}(\beta_1 r)+2A_{12m}W_{m+1}(\beta_1 r);$$
$$\Psi_{r2}(r)+\Psi_{\theta 2}(r)=2A_{15m}Z_m(\beta_1 r)+2A_{16m}W_{m+1}(\beta_1 r), \quad (11)$$

Based on the assumption given in [3]:

$$\Psi_{r1}=-\Psi_{\theta 1}, \Psi_{r2}=-\Psi_{\theta 2}.$$

To determine arbitrary constants from$A_{1m}, \ldots, A_{16m}$ (11), we use contact and boundary conditions (2).

Then

$$\sigma_{rrn}=\bar{\lambda}_n\left(\frac{\partial u_{rn}}{\partial r}+\frac{\partial u_{zn}}{\partial r}+\frac{1}{r}\frac{\partial u_{\theta n}}{\partial\theta}+\frac{u_{rn}}{r}\right)+2\bar{\mu}_n\frac{\partial U_{rn}}{\partial r}; \quad (12)$$
$$\sigma_{r\theta n}=\mu_n\left(\frac{1}{r}\frac{\partial u_{rn}}{\partial\theta}+\frac{\partial u_{\theta n}}{\partial r}-\frac{u_{\theta n}}{r}\right); \quad (13)$$
$$\sigma_{rzn}=\mu_n\left(\frac{\partial u_{rn}}{\partial z}+\frac{\partial u_{\theta n}}{\partial r}\right). \quad 14)$$

Substituting (11) into the functions (12)-(14), and taking into account the relations (3), (5) and (6), we obtain:

$$\sigma_{rrn} = \left\{-\bar{\lambda}_n(\alpha_n^2+\gamma_p^2)\phi_n + 2\bar{\mu}_n\left[\phi_n^{''} + \frac{k}{r}(\Psi_n^{'} - \frac{1}{r}\Psi_{zn}) + \gamma_p\Psi_{zn}^{'}\right]\right\}cos(n\theta)e^{-i(-\omega t+\gamma_p z)}, \tag{15}$$

$$\sigma_{r\theta n} = \bar{\mu}_n\left[\frac{2k}{r}\left(\frac{1}{r}\phi_n - \phi_n^{'}\right) + (\beta_p^2\Psi_{zn}^{'} - 2\Psi_{zn}) + \gamma_p\left(\Psi_{rn}^{'} - \frac{k+1}{r}\Psi_{rn}\right)\right]sin(n\theta)e^{-i(-\omega t+\gamma_p z)}, \tag{16}$$

$$\sigma_{rzn} = \bar{\mu}_n\left[-2\gamma_p\phi_n^{'} - \frac{k\gamma_p}{r}\Psi_{zn} - \frac{k}{r}\Psi_{zn}^{'} - (\frac{k^2-k-1}{r^2} - \beta_n^2 + \gamma_p^2)\Psi_{zn}\right]cos(n\theta)e^{-i^{(-\omega t+\gamma_p z)}}. \tag{17}$$

Substituting (11), (15) - (16) in condition (2), we get the system of equations. From the condition of existence of a nontrivial solution of a system of homogeneous algebraic equations, we obtain equation

$$|c_{ij}| = 0$$

where $c_{ij}(i,j = 1,2..,12)$ are the coefficients determining the dispersion equation, calculated as follows:

$$c_{1,1} = Z_m^{'}(\alpha_1 r_2); c_{1,2} = W_m^{'}(\alpha_1 r_2); c_{1,3} = \gamma_p Z_{m+1}(\beta_1 r_2);$$
$$c_{1,4} = \gamma_p W_{m+1}(\beta_1 r_2); c_{1,5} = (k/r_2)Z_m(\beta_1 r_2);$$
$$c_{1,6} = (k/r_2)W_m(\beta_1 r_2); c_{1,7} = -Z_n^{'}(\alpha_2 r_2);$$
$$c_{1,8} = -W_n^{'}(\alpha_2 r_2); c_{1,9} = -\gamma Z_{n+1}(\beta_2 r_2);$$
$$c_{1,10} = -\gamma_p W_{n+1}(\beta_2 r); c_{1,11} = -(k/r_2)Z_n(\beta_2 r);$$
$$c_{1,12} = -(k/r_2)Z_n(\beta_2 r);$$
$$c_{2,1} = -(k/r_2)Z_m(\alpha_1 r_2); c_{2,2} = -(k/r_2)W_m(\alpha_1 r_2);$$
$$c_{2,3} = \gamma_p Z_{m+1}(\beta_1 r_2); c_{2,4} = \gamma_p W_{m+1}(\beta_1 r_2);$$
$$c_{2,5} = -Z_{m+1}^{'}(\beta_1 r_2); c_{2,6} = -W_{m+1}^{'}(\beta_1 r_2);$$
$$c_{2,7} = (k/r_2)Z_m(\alpha_2 r_2); c_{2,8} = (k/r_2)W_m(\alpha_2 r_2);$$
$$c_{2,9} = -\gamma_p Z_{m+1}(\beta_2 r_2); c_{2,10} = -\gamma_p W_{m+1}(\beta_2 r_2);$$
$$c_{2,11} = Z_m^{'}(\beta_2 r_2); c_{2,12} = W_m^{'}(\beta_2 r_2);$$
$$c_{3,1} = -\gamma_p Z_m(\alpha_1 r_2); a_{3,2} = -\gamma_p W_m(\alpha_1 r_2);$$
$$c_{3,3} = -[(k+1)/r_2]Z_{m+1}(\beta_1 r_2) - Z_{m+1}^{'}(\beta_1 r_2);$$
$$c_{3,4} = -[(k+1)/r_2]W_{m+1}(\beta_1 r_2) - W_{m+1}^{'}(\beta_1 r_2);$$
$$c_{3,7} = \gamma_p Z_m(\alpha_2 r_2); a_{3,8} = \gamma_p W_m(\alpha_2 r);$$
$$c_{3,9} = [(k+1)/r_2]Z_{m+1}(\beta_2 r_2) + Z_{m+1}^{'}(\beta_2 r_2);$$
$$c_{3,10} = [(k+1)/r_2]W_{m+1}(\beta_2 r_2) + W_{m+1}^{'}(\beta_2 r_2);$$
$$c_{3,11} = 0; a_{3,12} = 0; a_{3,6} = 0; a_{3,5} = 0;$$
$$c_{4,1} = -\bar{\lambda}_1[\alpha_1^2+\gamma_p^2]Z_m(\alpha_1 r_2) + 2\bar{\mu}_1 Z_m^{''}(\alpha_1 r);$$
$$c_{4,2} = -\bar{\lambda}_1[\alpha_1^2+\gamma_p^2]W_m(\alpha_1 r) + 2\bar{\mu}_1 W_m^{''}(\alpha_1 r);$$
$$c_{4,3} = 2\bar{\mu}_1\gamma_p Z_{m+1}^{'}(\beta_1 r_2); a_{4,4} = 2\bar{\mu}_1\gamma_p W_{m+1}^{'}(\beta_1 r_2);$$
$$c_{4,5} = 2\bar{\mu}_1(k/r_2)\left[Z_{m+1}^{'}(\beta_1 r_2) - (1/r_2)Z_m(\beta_1 r_2)\right];$$
$$c_{4,6} = 2\bar{\mu}_1(k/r_2)\left[W_{m+1}^{'}(\beta_1 r_2) - (1/r_2)W_m(\beta_1 r_2)\right];$$
$$c_{4,7} = \bar{\lambda}_2[\alpha_2^2+\gamma_p^2]Z_m(\alpha_2 r_2) - 2\bar{\mu}_2 Z_m^{''}(\alpha_2 r_2);$$
$$c_{4,8} = \bar{\lambda}_2[\alpha_2^2+\gamma_p^2]W_m(\alpha_2 r_2) + 2\bar{\mu}_2 W_m^{''}(\alpha_2 r_2);$$
$$c_{4,9} = -2\bar{\mu}_2\gamma_p W_{m+1}^{'}(\beta_2 r_2); c_{4,10} = -2\bar{\mu}_2\gamma_p W_{m+1}^{'}(\beta_2 r_2);$$
$$c_{4,11} = -2\bar{\mu}_2(k/r_2)\left[Z_{m+1}^{'}(\beta_2 r_2) - (1/r_2)Z_{m+1}(\beta_2 r_2)\right];$$
$$c_{4,11} = -2\bar{\mu}_2(k/r_2)\left[W_{m+1}^{'}(\beta_2 r_2) - (1/r_2)W_{m+1}(\beta_2 r_2)\right];$$
$$c_{5,1} = 2\bar{\mu}_1(k/r_2)\left[(1/r_2)Z_m(\alpha_1 r_2) - Z_m^{'}(\alpha_1 r_2)\right];$$

$$c_{5,2} = 2\bar{\mu}_1(k/r_2)\left[(1/r_2)W_m(\alpha_1 r_2) - W_m^{'}(\alpha_1 r_2)\right];$$
$$c_{5,3} = \bar{\mu}_1\gamma_p(k/r_2)\left[Z_{m+1}^{'}(\beta_1 r_2) - [(k+1)/r_2]Z_{m+1}(\beta_1 r_2)\right];$$
$$c_{5,4} = \bar{\mu}_1\gamma_p(k/r_2)\left[W_{m+1}^{'}(\beta_1 r_2) - [(k+1)/r_2]W_{m+1}(\beta_1 r_2)\right];$$
$$c_{5,5} = \bar{\mu}_1\left[\beta_1^2 Z_m(\beta_1 r_2) - Z_m^{'}(\beta_1 r_2)\right];$$
$$c_{5,6} = \bar{\mu}_1\left[\beta_1^2 W_m(\beta_1 r_2) - W_m^{'}(\beta_1 r_2)\right];$$
$$c_{5,7} = -2\bar{\mu}_2(k/r_2)\left[(1/r_2)Z_m(\alpha_2 r_2) - Z_m^{'}(\alpha_2 r_2)\right];$$
$$c_{5,8} = -2\bar{\mu}_2(k/r_2)\left[(1/r_2)W_m(\alpha_2 r_2) - W_m^{'}(\alpha_2 r_2)\right];$$
$$c_{5,9} = -\bar{\mu}_2\gamma_p\left[Z_{m+1}^{'}(\beta_2 r_2) - [(k+1)/r_2]Z_{m+1}(\beta_2 r_2)\right];$$
$$c_{5,10} = \bar{\mu}_2\gamma_p\left[W_{m+1}^{'}(\beta_2 r_2) - [(k+1)/r_2]W_{m+1}(\beta_2 r_2)\right];$$
$$c_{5,11} = \bar{\mu}_2\left[\beta_2^2 Z_m(\beta_2 r_2) - Z_m^{'}(\beta_2 r_2)\right];$$
$$c_{5,12} = \bar{\mu}_2\left[\beta_2^2 W_m(\beta_{IIr2}) - W_m^{'}(\beta_{IIr2})\right];$$
$$c_{6,1} = -2\bar{\mu}_1\gamma_p\left[Z_m^{'}(\alpha_1 r_2)\right];$$
$$c_{6,2} = -2\bar{\mu}_1\gamma_p\left[W_m^{'}(\alpha_1 r)\right];$$
$$c_{6,3} = -\bar{\mu}_1((k/r_2)Z_{m+1}^{'}(\beta_1 r_2) + \{[(k^2 - k - 1/r_2^2)] - \beta_1^2 + \gamma_p^2\}Z_{m+1}^{'}(\beta_1 r_2));$$
$$c_{6,4} = -\bar{\mu}_1((k/r_2)W_{m+1}^{'}(\beta_1 r_2) + \{[(k^2 - k - 1/r_2^2)] - \beta_1^2 + \gamma_p^2\}W_{m+1}^{'}(\beta_1 r_2));$$
$$c_{6,5} = -\bar{\mu}_1(k\gamma_p/r_2)Z_m(\beta_1 r_2);\ c_{6,6} = -\bar{\mu}_1(k\gamma_p/r_2)W_m(\beta_1 r_2);$$
$$c_{6,7} = 2\bar{\mu}_2\gamma_p(k\gamma_p/r_2)Z_m^{'}(\alpha_2 r_2);\ c_{6,8} = 2\bar{\mu}_2\gamma_p(k\gamma/r_2)W_m^{'}(\alpha_2 r_2);$$
$$c_{6,9} = \bar{\mu}_2((k/r_2)Z_{m+1}^{'}(\beta_2 r_2) + \{[(k^2 - k - 1/r_2^2)] - \beta_2^2 + \gamma_p^2\}Z_{m+1}(\beta_2 r_2));$$
$$c_{6,10} = \bar{\mu}_2((k/r_2)W_{m+1}^{'}(\beta_2 r_2) + \{[(k^2 - k - 1/r_2^2)] - \beta_2^2 + \gamma_p^2\}W_{m+1}(\beta_2 r_2)),$$
$$c_{6,11} = \bar{\mu}_2\gamma_p(k\gamma_p/r_2)Z_m(\beta_2 r_2);$$
$$c_{6,12} = \bar{\mu}_2\gamma_p(k\gamma_p/r_2)W_m(\beta_2 r_2).$$

The other elements of the main determinant are defined similarly.

3 Results and analysis

After simple transformations, we obtain a dispersion equation, which is solved numerically by the Muller method on complex arithmetic.

To obtain numerical results for a two-layer viscoelastic pipe, the following parameter values are taken:

Dт=50mm, Dи=5mm, the outer pipe is made of X12 steel: $\mu_T = 80 \cdot 10^9 Pa, \rho_I = 7800 kg/m^3$. the inner pipe is made of 30L steel: $\mu_T = 70 \cdot 10^9 Pa, \rho_I = 7500 kg/m^3$.

As an example of a viscoelastic material, we take the Koltunov - Rzhanitsyn relaxation core: $R_к(t) = A_к e^{-\beta_к t}/t^{1-\alpha_к}$, with parameters: $A_к = 0{,}048$; $\beta_к = 0{,}05$; $\alpha_к = 0{,}1$ (k=1,2). Figure 2 shows: 1 and 3 - for a homogeneous pipe (excluding dispersion), 2,4,5 – for two-layer homogeneous pipes.

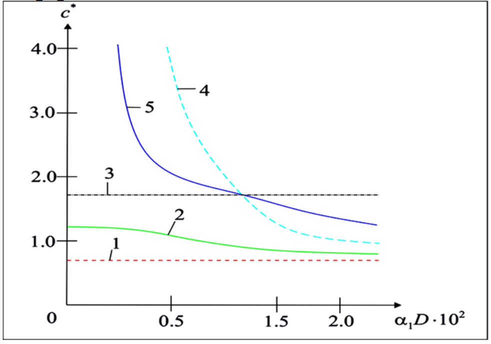

Fig. 2. Change of phase velocity depending on the wave number.

Figures 3 and 4 show the changing real part of the phase velocity (six modes) depending on the wave number at different ratios of radii ($R_1 = r_1/r_3, R_2 = r_2/r_3, R_3 = 1$).

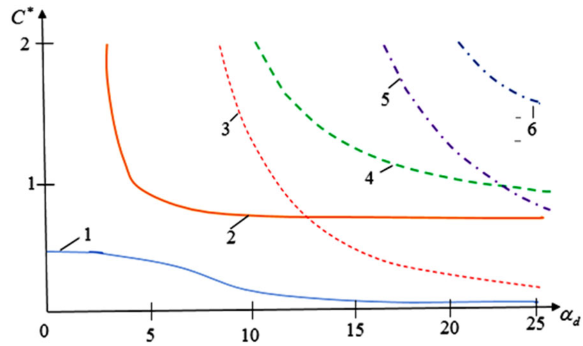

Fig. 3. Change of phase velocity depending on the wave number.

It can be seen that with an increase in the wave number, the real parts of the phase velocity are killed and approach the asymptotic. The nature of the change in the imaginary part of the phase velocity of all modes for viscoelastic cylindrical bodies is almost the same (Figure 3 and Figure 4).

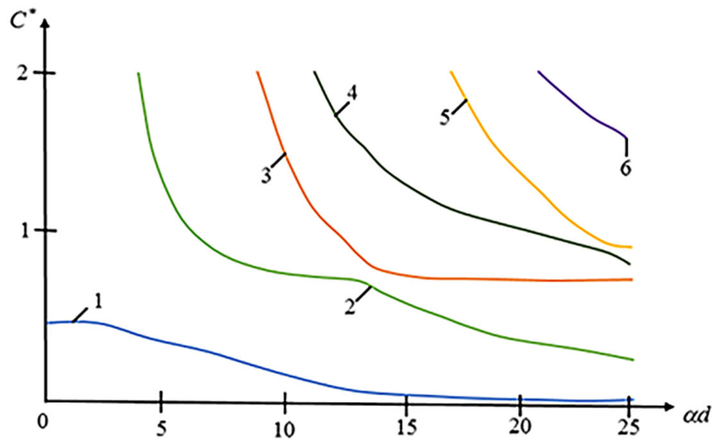

Fig. 4. Change of the phase velocity depending on the wave number.

4 Conclusions

A structurally inhomogeneous mechanical system is investigated for various geometric and physico-mechanical parameters of the mechanical system. Based on the above study, it was found that the real parts of the wave velocity will increase by only a 5%, and the imaginary parts - have changed radically. So that, phase velocities with an increase in the number of waves along the circumference of the cylinder first decrease, and then begin to increase.

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