Transient analysis of electrical drives:

Applications to the permanent magnet synchronous generator

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Abstract. This paper establishes that *differential flatness*, a property shared by most of the model of electrical drives, allows to perform transient analysis on the models of electrical machines, and especially electric generators which are so important in renewable energy systems (wind turbine, tidal turbine...). Transient analysis of electrical machines is indeed a difficult subject not completely solved today. The possibility to parameterize the trajectories of a system from the differential flatness property gives the possibility to perform simple transient analysis in various operation modes. The example of the Permanent Magnet Synchronous Generator (PMSG) illustrates the paper.

1 Introduction

Differential flatness has been introduced in 1992 and has permited to develop efficient and robust control laws for nonlinear systems [1, 2]. Numerous examples have been addressed and will not be recalled here.

Since its introduction, flatness has been intimately linked to feedback linearization which is also an inportant field in control. In order to emphasis the trajectory aspect of differential flatness, the concept of *exact feedforward linearization based on differential flatness* has been recently introduced [3, 4]. This allows to give general justifications of the stability and robustness of flatness-based control laws [4, 5].

The aim of the present work is to show that exact feedforward linearization based on differential flatness is also a powerful tool in model dynamical analysis. It is illustrated by the presentation of the analyze of transient operation of electrical generators. The paper is illusrated with the permanent magnet synchronous generator.

The paper is organized as follows: Sec. 2 exposes the basic notions from differential flatness; Sec. 3 shortly recalls the notion of exact feedforward linearization based on differential flatness; Sec 4 presents the model of the permanent magnet synchronous generator (PMSG) used for illustration; Sec. 5 contains the analysis of various operation modes of the generator using exact feedforward linearization based on differential flatness; Sec. 6 concludes the paper.

2 Flatness

2.1 Recall

Consider a nonlinear control system with *input* $e = (e_1, \ldots, e_l)$ and *state* $x = (x_1, \ldots, x_n)$ given by the corresponding state-variable representation:

$$\dot{x} = f(x, e) \tag{1}$$

where f is a smooth vector field parameterized by the input e. Note that the input can involve both control and disturbance $e = (u, \varpi)$ where $u = (u_1, \ldots, u_m)$ is the *control input* and $\varpi = (\varpi_1, \ldots, \varpi_q)$ is the *disturbance input* (of course, $m + q = l$). Note also that in some cases, f can depend on time but we choose to avoid the explicit reference to the time for notational convenience.

System (1) is said to be *differentially flat* [2] if there exists a set of l variables $z = (z_1, \ldots, z_l)$ having the following three properties:

1. The components of z are differentially independent, that is there is no differential equation relating them or any of their derivatives;

2. The components of z can be expressed in terms of the system variables, that is :

$$z = h\left(x, e, \dot{e}, \ldots, e^{(\alpha)}\right) \tag{2}$$

3. The variables of the system can be expressed in terms of the flat output and its derivatives, that is:

$$e = A\left(z, \dot{z}, \ldots, z^{(\beta)}\right) \tag{3}$$

$$x = B\left(z, \dot{z}, \ldots, z^{(\gamma)}\right) \tag{4}$$

Notice that l is precisely the number of input components. Such a set of variables $z = (z_1, \ldots, z_l)$ is called a *flat output* or *linearizing output* of the system (1).

Notice that the second condition means, that the components of a flat output are variables of the system (1) or can be simply expressed in terms of the system variables.

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The third condition implies that every variable of the system or every function of the variables of (1) can be calculated using the flat output and its derivatives, without the need for integration of any differential equation. Finally, the first condition guarantees that the components of the flat output are not related by any differential equation; they are completely free and therefore it is possible to impress their trajectories (predicted trajectories) in order to achieve the control of (1).

2.2 Parameterization of trajectories

Consider a time interval $I \subset \mathbb{R}$ of the form $I = [t_o, t_f)$ with $t_o \in \mathbb{R}$, $t_f > t_o$ and where t_f is either a real or $t_f = +\infty$. A function of time[1]

$$z^\sharp : I \longrightarrow \mathbb{R}^l \tag{5}$$

is said to be an *(admissible) nominal trajectory* for the flat output z of (1) if it is sufficiently smooth in order that expressions (3) and (4) are defined everywhere on the interval I. This implies in particular that $z^\sharp$ avoids any eventual singularity of the functions A and B involved in (3) and (4). In some cases, an admissible nominal trajectory can be called an *(admissible) reference trajectory* or even an *(admissible) predicted trajectory* if one wants to emphasis the predictive aspect of any flatness based control.

For any admissible nominal trajectory $z^\sharp$ of the flat output we obtain a *nominal input trajectory* $e^\sharp$ of the input e defined by

$$e^\sharp(t) = A\left(z^\sharp(t), \dot{z}^\sharp(t), \ldots, z^{\sharp(\beta)}(t)\right), \quad \forall t \in I \tag{6}$$

or, more precisely, component by component

$$\left\{\begin{array}{rcl} e_1^\sharp(t) & = & A_1\left(z^\sharp(t), \dot{z}^\sharp(t), \ldots, z^{\sharp(\beta_1)}(t)\right) \\ & \vdots & \\ e_l^\sharp(t) & = & A_l\left(z^\sharp(t), \dot{z}^\sharp(t), \ldots, z^{\sharp(\beta_l)}(t)\right) \end{array}\right. , \ \forall t \in I$$

and the index β appearing in (3) and (6) satisfies $\beta = \max\{\beta_1, \ldots, \beta_l\}$.

In the same manner, any (admissible) nominal trajectory $z^\sharp$ of the flat output gives rise to a *nominal state trajectory* $x^\sharp$ of the state x which is defined by

$$x^\sharp(t) = B\left(z^\sharp(t), \dot{z}^\sharp(t), \ldots, z^{\sharp(\gamma)}(t)\right), \quad \forall t \in I \tag{7}$$

which can also be written componentwise as:

$$\left\{\begin{array}{rcl} x_1^\sharp(t) & = & B_1\left(z^\sharp(t), \dot{z}^\sharp(t), \ldots, z^{\sharp(\gamma_1)}(t)\right) \\ & \vdots & \\ x_n^\sharp(t) & = & B_n\left(z^\sharp(t), \dot{z}^\sharp(t), \ldots, z^{\sharp(\gamma_n)}(t)\right) \end{array}\right. , \ \forall t \in I$$

and, as expected, $\gamma = \max\{\gamma_1, \ldots, \gamma_n\}$.

Equations (6) and (7) illustrate the *parameterization of the trajectories* $t \mapsto (e^\sharp(t), x^\sharp(t))$ of the differential system (1) in terms of the flat output. In some case, $e^\sharp$ is called the *reference input trajectory* or even the *predicted input trajectory* of the input e while $x^\sharp$ is called the *reference* or *predicted state trajectory* of the state x. In order not to unnecessarily weight down the vocabulary and notations we choose not to mention that input or state reference trajectories depend on the choice of admissible flat output trajectories.

Notice that $z^\sharp$ is a nominal trajectory, that is a time function that can be chosen. Consequently, in Eqns. (6) and (7) there is no problem with eventual noisy signal that must be differentiated.

3 Exact feedforward linearization based on differential flatness

A natural question that arises in the context of differential flatness is what is the behavior of system (1) when one applies to it the nominal input $e^\sharp$ deduced by (6) from a nominal flat output trajectory $z^\sharp$. To answer precisely this question and to study other related problems we need to introduce a "controller-like form" obtained by change of state and a related "Brunovský-like form".

We are now able to recall the important result of [4] which establish the property of *exact feedforward linearization*, that is the answer of the previous question: What happens when applying a nominal control to a flat system.

Theorem 1 *The application to system* (1) *of the nominal input $e^\sharp$, defined by* (6)*, from the corresponding consistent initial condition $x^\sharp(t_o)$ generates a trajectory $\tilde{x}$ of* (1) *which exists on* $[t_o, t_f)$ *and corresponds to the trajectory generated by a linear system under Brunovský form driven with appropriate derivatives of the flat output nominal trajectory.*

This theorem originally announced in [3] was first presented as an equivalence result: A flat system is equivalent *by a diffeomorphism* to a linear Brunovský form *on nominal trajectories*. Note that the underlying notion of equivalence is different to the one presented in the original works on differential flatness [1, 6]. In the latter, it was established that flat systems were *(endogenous) feedback equivalent* to linear controllable ones. In the present work this is a (local) diffeomorphic equivalence on nominal trajectories.

In the case that the initial condition is not consistent with the nominal trajectory, we are able to prove the existence on I (at least for a finite length interval) of a solution which lies in the neighborhood of the trajectory of the Brunnovský system.

4 Model of the permanent magnet synchronous generator

This section is devoted to the exposition of the model of the permanent magnet synchronous generator (PMSG) and its flatness property. According to [7] the model of the equations corresponding to the behavior of this electrical generator can be written as follows. For the sake of

[1] In some cases, the models are written in a complex framework and in this case, we use a complex-valuated function $z^\sharp : I \longrightarrow \mathbb{C}^l$ without further difficulty.

simplicity we consider only a two-phases machine but everything applies to the most common case of three-phases generator using usual transformation of coordinates.

We follow the standard hypothesis of modeling of this generator, namely: the magnetic systems is linear, the open-circuit voltages induced by rotating the permanent magnet rotor at constant speed are sinusoidal, large stator currents can be tolerated without significant demagnetization of the permanent magnets, damper windings are not considered. Moreover, we assume that the air gap is uniform, which implies that the the mutual inductance between the two stator windings is equal to zero.

The stator voltage equations reads:

$$v_a = R_s i_a + \frac{d\psi_a}{dt} \tag{8a}$$
$$v_b = R_s i_b + \frac{d\psi_b}{dt} \tag{8b}$$

where v_a, v_b are the voltages supplied to the two phases; i_a, i_b are the currents; ψ_a, ψ_b are the fluxes of the two windings. The fluxes can be expressed in terms of the stator currents and the magnitude of the flux of the permanent magnets ψ_m as follows:

$$\psi_a = L_s i_a + \psi_m \sin(n_p\theta) \tag{9a}$$
$$\psi_b = L_s i_b - \psi_m \cos(n_p\theta) \tag{9b}$$

where θ stands for the rotor angular position and n_p is the number of pole pairs.

The mechanical equation is simply

$$J\ddot{\theta} = T_e - T_l \tag{10}$$

where T_l is the motor torque applied to the shaft and T_e is to electromagnetic torque, the expression of which is:

$$T_e = n_p\psi_m \left[i_a \cos(n_p\theta) + i_b \sin(n_p\theta) \right] \tag{11}$$

As usual, it is useful to consider the Park transformation of the electrical variable in order to simplify the model [7]. This invertible transformation is defined by

$$\begin{pmatrix} f_d \\ f_q \end{pmatrix} = \begin{pmatrix} \sin(n_p\theta) & -\cos(n_p\theta) \\ \cos(n_p\theta) & \sin(n_p\theta) \end{pmatrix} \begin{pmatrix} f_a \\ f_b \end{pmatrix} \tag{12}$$

where "f" stands either for "ψ", "i", or "v". The indices "d" and "q" mean "direct" and "quadrature". After the transformation the equations (8), (9), and (11) become respectively:

$$v_d = R_s i_d + \frac{d\psi_d}{dt} - n_p\dot{\theta}\psi_q \tag{13a}$$
$$v_q = R_s i_q + \frac{d\psi_q}{dt} + n_p\dot{\theta}\psi_d \tag{13b}$$

$$\psi_d = L_s i_d + \psi_m \tag{14a}$$
$$\psi_q = L_s i_q \tag{14b}$$

and

$$T_e = n_p\psi_m i_q \tag{15}$$

The input of the PMSM is $u = (v_d, v_q)^t$ (this is $(v_a, v_b)^t$ in original coordinates) and $\varpi = (T_l)$. A state-space model can be written by choosing $x = (\theta, \Omega, i_d, i_q)^t$ where Ω is the angular speed of the shaft:

$$\dot{\theta} = \Omega \tag{16a}$$
$$\dot{\Omega} = \frac{1}{J}\left(n_p\psi_m i_q - T_l \right) \tag{16b}$$
$$\frac{di_d}{dt} = \frac{1}{L_s}\left(-R_s i_d + n_p L_s \Omega i_q + v_d \right) \tag{16c}$$
$$\frac{di_d}{dt} = \frac{1}{L_s}\left(-R_s i_q - n_p L_s \Omega i_d - n_p\psi_m\Omega + v_q \right) \tag{16d}$$

The model of the PMSM is flat with flat output $z = (\theta, i_d, T_l)$. This can be simply checked as follows:

$$\Omega = \dot{\theta} \tag{17a}$$
$$i_q = \frac{J\ddot{\theta} + T_l}{n_p\psi_m} \tag{17b}$$
$$\begin{aligned} v_q &= R_s i_q + L_s\frac{di_q}{dt} + n_p L_s\dot{\theta} i_d + n_p\psi_m\dot{\theta} \\ &= R_s\frac{J\ddot{\theta} + T_l}{n_p\psi_m} + L_s\frac{J\theta^{(3)} + \dot{T}_l}{n_p\psi_m} + n_p L_s\dot{\theta} i_d + n_p\psi_m\dot{\theta} \end{aligned} \tag{17c}$$
$$\begin{aligned} v_d &= R_s i_d + L_s\frac{di_d}{dt} - n_p L_s\dot{\theta} i_q \\ &= R_s i_d + L_s\frac{di_d}{dt} - L_s\dot{\theta}\frac{J\ddot{\theta} + T_l}{\psi_m} \end{aligned} \tag{17d}$$

Finally note that the original control input v_a and v_b can be explicited by plugging eqns. (17c) and (17d) in the expressions of v_a and v_b in terms of v_d and v_q:

$$v_a = v_d \sin(n_p\theta) + v_q \cos(n_p\theta) \tag{18a}$$
$$v_b = -v_d \cos(n_p\theta) + v_q \sin(n_p\theta) \tag{18b}$$

5 Analysis of various operations of the PMSG

5.1 Steady-state operation

The constant speed operation under constant load can be easily analyzed from flatness: The steady state is simply defined, on $[t_o, +\infty[$, by

$$\dot{\theta} = \Omega_o \tag{19a}$$
$$T_l = T_{lo} \tag{19b}$$
$$i_d = 0 \tag{19c}$$

where Ω_o and T_{lo} are constant. Eqn. (19) defines an admissible trajectory of the flat output. Consequently, we obtain for (17):

$$\Omega^o = \Omega_o \tag{20a}$$
$$i_q^o = \frac{T_{lo}}{n_p\psi_m} \tag{20b}$$
$$v_q^o = R_s\frac{T_{lo}}{n_p\psi_m} + n_p\psi_m\Omega_o \tag{20c}$$
$$v_d^o = -\frac{L_s}{\psi_m}\Omega_o T_{lo} \tag{20d}$$

where Ω^o, i_q^o, v_q^o, and v_d^o denote the steady values of Ω, i_q, v_q, and v_d. As expected the supply voltages v_a and v_b must be of the form:

$$\tilde{v_a}(t) = V_o \cos(\omega_o t + \phi_o) \tag{21a}$$
$$\tilde{v_b}(t) = V_o \sin(\omega_o t + \phi_o) \tag{21b}$$

and the corresponding windings current are sinusoidal too:

$$\tilde{i_a}(t) = I_o \cos(\omega_o t + \varphi_o) \tag{22a}$$
$$\tilde{i_b}(t) = I_o \sin(\omega_o t + \varphi_o) \tag{22b}$$

The problem that arises in classical analysis of steady state operation is that there is no precise analysis of the behavior of the system when on applies (21) from rest. This require a transient analysis and the best that can be achieved is to show that the steady-state operation defined by (20) is only reached asymptotically.

$$\begin{aligned}\lim_{t\to\infty} \Omega(t) &= \Omega^o \\ \lim_{t\to\infty} i_q(t) &= i_q^o \\ \lim_{t\to\infty} i_d(t) &= 0\end{aligned}$$

Note that the two last equations also means that the windings currents i_a and i_b are asymptotically sinusoidal. To our best knowledge, this has not been established anywhere precisely but it is always assumed to be true. Fortunately, things are working very well from the passivity property of the generators. The proof of this fact can be carried out by application of Kelemen's theorem [8].

5.2 Transient operations

From flatness we can analyze also various transient operations of the generator and even define a particular transient in order to reach, in finite time, the expected steady state defined in (19) and (20). This is done simply by defining reference trajectories of the flat output that are consistent with initial conditions. The trajectory of the control is then defined by (17c) and (17d). Theorem 1 ensures that the systems evolves on the reference trajectory of the flat output.

From (17c) and (17d) we see that the expressions of v_d and v_q are defined if the third derivative of θ, the first derivative of T_l, and the first derivative of i_d are piecewise-regulated [9]. That is $\theta^{(3)}$, $\frac{di_d}{dt}$ and $\dot{T}_l$ admit left and right limits everywhere. This can be achieve by selecting appropriate spline functions. Consider a sequence of time times $t_0 < t_1 < \cdots t_k < t_{k+1} < \cdots < t_N$. The trajectories of the components of the flat output are simply defined by polynomials expressions in each interval (t_k, t_{k+1}):

$$\theta(t) = \sum_{j=0}^{n_1} \theta^{k,j} (t - t_k)^j \tag{23a}$$

$$i_d(t) = \sum_{j=0}^{n_2} {i_d}^{k,j} (t - t_k)^j \tag{23b}$$

$$T_l(t) = \sum_{j=0}^{n_3} {T_l}^{k,j} (t - t_k)^j \tag{23c}$$

where the $\theta^{k,j}$'s, ${i_d}^{k,j}$'s, and ${T_l}^{k,j}$'s are real constants. The regularity assumption on θ, i_d and T_l is easily achieved by impressing continuity constraints on appropriate derivatives of θ, i_d, and T_l.

From (17), (18) and (23) it is possible to obtain the analytic expressions of all the variables in any operation mode.

As an example, we can consider the operation of the generator defined by:

- $t \in [0; 0.1\,\mathrm{s}[$, acceleration from rest;
- $t \in [0.1\,\mathrm{s}; 0.3\,\mathrm{s}[$, constant speed operation;
- $t \in [0.3\,\mathrm{s}; 0.4\,\mathrm{s}[$, deceleration to stall.

Fig. 1 reports the curves in the case for which there is no load ($T_l \equiv 0$), while in the case of Fig. 2 a constant load torque is applied to the generator between $t = 0.2$ s and $t = 0.3$ s. The direct current $i_d = 0$ is kept to 0 as it is usually done.

As expected, we see that the supply voltages are sinusoidal during the time interval of constant speed under constant torque. During the two transients (acceleration and breaking) the voltage are not sinusoidal and the flatness property allows to express their time evolution with no difficulty.

The curves of Figs. 1 and 2 are not obtained by simulation of the model, but simply by performing the calculations derived from the flatness property (eqns. (17) and (18)) and an appropriate choice of the flat output trajectory in order to achieve a particular transient operation of the machine. Theorem 1 ensures that a simultation with the obtained supply voltage, with the right initial condition, will lead to the same curves.

As announced before, the generator can reach the usual steady state in finite time. The transient need to reach this steady state is more complicated than simple sine signals for the inputs.

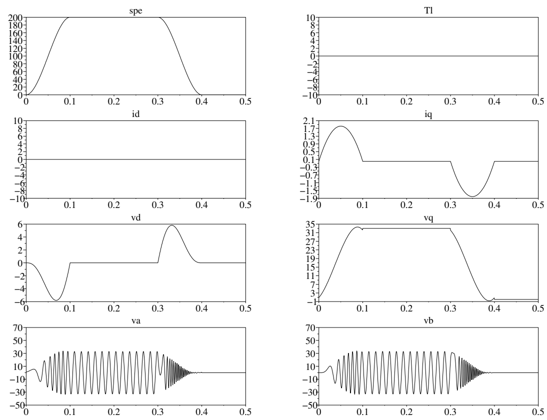

Figure 1. Unloaded transient operation

6 Conclusion

We have exposed the fact that differential flatness is not only important for control purpose but also for analyzing models of control systems. The same study can be carried

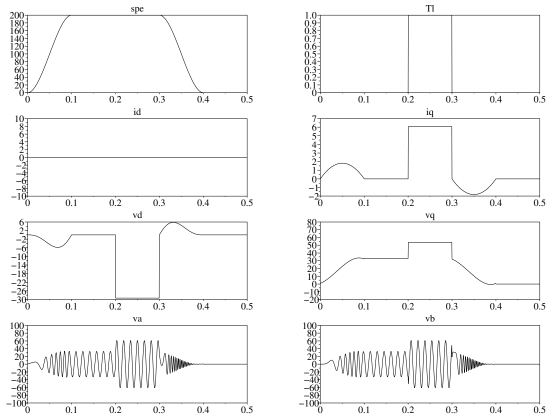

Figure 2. Loaded transient operation

out to all flat systems and, in particular, to other kind of electrical machines which are flat [10–15]. In our opinion, this fact should be taken into account to give a new light in the teaching of electrical machines in transient operations.

References

[1] M. Fliess, J. Lévine, P. Martin, P. Rouchon, Internat. J. Control **61**, 1327 (1995)

[2] M. Fliess, J. Lévine, P. Martin, P. Rouchon, IEEE Trans. Automat. Contr. **44**, 922 (1999)

[3] V. Hagenmeyer, E. Delaleau, in *Nonlinear and Adaptive Control (NCN4 2001)*, edited by A. Zinober, D. Owens (Springer-Verlag, Berlin Heidelberg, 2003), Vol. 281 of *Lect. Notes Control. Inform. Sci.*, pp. 161–170

[4] V. Hagenmeyer, E. Delaleau, Internat. J. Control **76**, 537 (2003)

[5] V. Hagenmeyer, E. Delaleau, Automatica J. IFAC **39**, 1941 (2003)

[6] M. Fliess, J. Lévine, P. Martin, P. Rouchon, C. R. Acad. Sci. Paris Sér. I Math. **315**, 619 (1992)

[7] P.C. Krause, O. Wasynczuk, *Electromechanical Motion Devices* (McGraw-Hill Book company, New York, 1989)

[8] M. Kelemen, IEEE Trans. Automat. Control **31**, 766 (1986)

[9] J. Dieudonné, *Foundations of Modern Analysis* (Academic Press, 1969)

[10] A. Chelouah, E. Delaleau, P. Martin, P. Rouchon, *Differential flatness and control of induction motors*, in *Proceedings of the Symposium on Control, Optimization and Supervision; Computational Engineering in Systems Applications IMACS Multiconference*, edited by P. Borne, M. Staroswiecki, J.P. Cassar, S. El Khattabi (Lille, France, 1996), pp. 80–85

[11] J. Boichot, E. Delaleau, N.V. Diep, J. Lévine, E. Parzy, *Modelling and control of a high-precision positionning system actuated by a linear synchronous motor*, in *Preprints of the 14th IFAC World Congress (CD-Rom)* (Beijing, Chine, 1999)

[12] E. Delaleau, J.P. Louis, R. Ortega, Int. J. Appl. Math. Comput. Sci. **11**, 105 (2001)

[13] V. Hagenmeyer, P. Kohlrausch, E. Delaleau, in *Nonlinear Control in the Year 2000 (Volume 1)*, edited by A. Isidori, F. Lamnabhi-Lagarrigue, W. Respondek (Springer-Verlag, London, 2001), Vol. 258 of *Lect. Notes Control. Inform. Sci.*, pp. 439–452

[14] V. Hagenmeyer, A. Ranftl, E. Delaleau, in *Nonlinear and Adaptive Control (NCN4 2001)*, edited by A. Zinober, D. Owens (Springer-Verlag, Berlin Heidelberg, 2003), Vol. 281 of *Lect. Notes Control. Inform. Sci.*, pp. 149–160

[15] E. Delaleau, E. Monmasson, J.P. Louis, *State-variable representation simulation and control of the doubly-fed induction generator*, in *Proceedigns 10th European Conference on Power Electronics and Applications (CD-Rom)* (Toulouse, France, 2003)