# Monte Carlo solution of heat conductivity problems with quadratic nonlinearity on the boundary of domain

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**Abstract.** In this work, we study the approach connected with branching Markov processes. Branching Markov random processes are created to solve boundary-value problems (BVR) with polynomial nonlinearities. The realization of these processes creates so called "trees". Unbiased estimators of the solution of the nonlinear problem are constructed on these random trees. We also calculated in parallel way the variance (statistical error) of the constructed unbiased estimators. We offer computational algorithms for solving some nonlinear diffusion problems which frequently appears in engineering problems, particularly in heat conductivity. The case of a quadratic nonlinearity for the boundary BVP is considered in detail. Algorithms, for diffusion BVPs with nonlinear boundary conditions, differ from proposed for linear diffusion BVPs algorithms early and we compare their efficiencies. The described algorithms applied to the computational problem of thermal engineering in the presence of a nonlinear boundary condition. Also, this problem is solved as a boundary value problem of conductive heat transfer.

## 1 Introduction

The application of Monte Carlo methods in various fields is constantly growing due to increases in computer capabilities. Increasing speed and memory, and the wide availability of multiprocessor computers, allows us to solve many problems using the "method of statistical sampling", better known as the Monte Carlo method.

Monte Carlo methods are known to have particular strengths. These include:

1. Algorithmic simplicity with a strong analogy to the underlying physical processes.
2. The ability to solve complex realistic problems that include sophisticated geometry and many physical processes.
3. The ability to solve problems set in very high dimensions.
4. The ability to obtain point solutions or a linear functional of the solution.
5. Error estimates can be empirically obtained for all types of problems.
6. Ease of efficient parallel implementation.

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In the current time three approaches are connected with solving nonlinear problems of engineering using method of Monte Carlo. The first approach implies the linearization of the problem (sequential solution of the series of linear problems). The second approach is connected with branching Markov processes. The third one is connected with optimal random search.

In this work, we study the approach connected with branching Markov processes. Branching Markov random processes are created to solve boundary-value problems (BVR) with polynomial nonlinearities. The realization of these processes creates so called “trees”. Unbiased estimators of the solution of the nonlinear problem are constructed on these random trees. We also calculated the variance (statistical error) of the constructed unbiased estimators.

We offer computational algorithms for solving some nonlinear diffusion problems which frequently appears in engineering. The case of a quadratic nonlinearity for the boundary BVP is considered in detail.

The solution of practical problem of mathematical physics, often in boundary conditions arise, which have polynomial nonlinearities. Let's consider one a problem (heat conductivity) of the following type

$$\frac{\partial u(x,t)}{\partial t} = a\Delta u(x,t) + af(x,t)/\lambda, \quad x \subset R^3, \tag{1}$$

where the temperature $u(x,t)$ is unknown, $t$ is time, $\Delta$ is the Laplace operator, $f(x,t)$ is a known function, $a$, $\lambda$ are temperature conductivity and heat conduction coefficients with boundary conditions

$$-\lambda \frac{\partial u(x_\Gamma,t)}{\partial n_{x_G}} = B(u^2(x_G,t) - u_{cp}^2), \tag{2}$$

where $x_G$ is a point on the boundary of our domain, $u_{cp} = const$ is a given temperature of the environment, $n_{x_G}$ is a direction of inner normal and initial conditions

$$u(x,0) = u_0(x), \tag{3}$$

where $u_0(x)$ is a known initial temperature.

The domain for this problem is bounded by two coaxial cylindrical surfaces with external and internal radiuses $R_1$, $R_2$ and length $L$. We denote this domain as $D$, and its boundary as G. We also assume that the material under consideration is isotropic relatively to heat transfer characteristics, and these characteristics do not depend on the temperature.

## 2 Methods

Boundary value problems with nonlinearities of this type cannot usually be solved analytically. That is way numerical approaches are used. A particularly, effective method for this problem is Monte Carlo method. Its application does not put limits on the configuration of the boundary allows one to calculate local values of the unknown function.

Let's consider using one of the above algorithms of solving the boundary value problem (1)-(3) by the Monte Carlo method. In a book [1] fundamental ideas for solving boundary value problems with nonlinear boundary conditions with the Monte Carlo method are connected to branching Markov processes. In addition, the algorithm of solving the grid analogue of the boundary problem (1)-(3) was given. Below we use the method of solving this problem by "walk on spheres", which is considered in detail in papers [2]. Random walk methods for Monte Carlo simulations of Brownian diffusion on sphere considered by many authors, for example [see.3]. Further development of this methods made by authors [4,5]. Nowadays, the efficiency of stochastic algorithms for solving the Dirichlet problem for the heat equation and algorithm walk on boundary method investigated by the authors [6,7,8]. For construction our algorithm we will transform the expression (2) into

$$\frac{\partial u(x_G,t)}{\partial n_{x_\Gamma}} = -cu^2(x_G,t) + A, \quad (4)$$

where $c = B/\lambda$ and $A = Bu_{cp}^2/\lambda$ for $u(x_\Gamma, t)$ we obtain

$$u(x_G,t) \approx p_2 u^2(x_l,t) + p_1 u(x_l,t) + p_0 A, \quad (5)$$

where $p_2 = cl/s,\ p_1 = 1/s,\ p_0 = l/s,\ s = 1 + l + cl,\ p_0 + p_1 + p_2 = 1.$

A random proses describes in a random walk of particle, which starts its random walk at $(x_0, 0)$, it moves to $(x_1, t_1)$, which is uniformly distributed on the sphere of maximal radius $r_1$ in the domain *D*. Then a new sphere will be constructed with center at $x_1$. This process (which called "random walk on spheres") continues until we reach to the boundary $G_\varepsilon$, the $\varepsilon$ - neighborhood of the boundary G.

## 2.1 Construction of stochastic algorithm

At every move of the particle to the next sphere, its weight increases by $f(x_i,t_i)r_i^2/c,$ where $i$ is step index. Also, equation (5) can be interpreted in the following way. When the particle reaches $G_\varepsilon$ at $x_1$ with probability $p_2$, the particle gives birth to two new equivalent particles and with probability $p_1$ one new particle. Also, with probability$p_0$, the random walk is absorbed and the weight becomes $A$. The particle continues its random walk analogous to the origin.

For each transition from point $(x_i, t_i)$ to point $(x_{i+1}, t_{i+1})$ the random time step $\tau_i$is sampled according to densities as described in the work [2]. The time is expressed in the form of $t_i = t_{i-1} + \tau_i$, where $t_0 = 0$, and $i = 1,2,$ .... The random walk process is terminated when either it is absorbed on the boundary, or the following condition holds

$$\sum_{i=1}^{k} t_i \geq T_0. \quad (6)$$

In the last case, the particle gets a weight, equal to initial condition $u_0(x_k)$.

When the initial particle gives birth at the *k*-th step, the current value of $t_k$is given to each of the new particles and we store these time values. After the next particle is absorbed, other particles (indexed by coordinates $x_i$ and time $t_i$), which have been saved earlier, are taken from memory for continuing processing. The process stopped when the branching Markov process is completely finished.

## 2.2 Numerical results

The validate of the method, we applied it to problem (1)-(3) with the following data:

$$L = 0.6m;\quad R_1 = 0.015m;\quad R_2 = 0.013m;\quad \lambda = 1.34Wt/(m\ degree);$$
$$a = 0.68 * 10^{-6}m^2/s;\quad V = 80Wt/m^2degree^2;\quad u_0 = 20^oC;$$
$$u_{average} = 100^oC;\quad f(x,t) = 0.26 * 10^8 Wt/m^3;\quad T_0 = 10.$$

A point estimator of the temperature was used at locating equidistant from the cylindrical surfaces and moving along the cylindrical axis. The origin of coordinates is the base of the cylinder. Results of the computation with a 97% confidence interval for a various value of $Z, \varepsilon$and $l$, using 10000 experiments in each series are given below:

**Table 1.** Numerical results of solution the problem (1)-(2)-(3).

| **Z, m** | **l, 10-2m** | **ε, 10-3m** | $\boldsymbol{u(x,t)},\ ^\circ\boldsymbol{C}$ | $\boldsymbol{3\hat{\sigma}},\ ^\circ\boldsymbol{C}$ |
|---|---|---|---|---|
| 0.57 | 0.1 | 0.3 | 86.71 | 2.15 |
| 0.57 | 0.1 | 0.1 | 89.34 | 2.77 |
| 0.3 | 0.06 | 0.1 | 87.14 | 2.14 |
| 0.3 | 0.1 | 0.3 | 86.32 | 2.93 |

Here $\widehat{\sigma}$ is standard deviation. The time dynamics the temperature of the point (0.0285m, 0.015 m, 0.3 m) is given in Table 2.

**Table 2.** Dynamics of changing of temperature at the point (0.0285, 0.015, 0.3).

| t | $u(x,t)$, $^{\circ}C$ | $3\widehat{\sigma}$, $^{\circ}C$ |
|---|---|---|
| 5 | 17.26 | 3.21 |
| 10 | 53.61 | 9.86 |
| 15 | 75.34 | 12.72 |
| 20 | 86.93 | 17.12 |

The CPU time for calculating the temperature was about 45 sec. on our PC.

## 3 Conclusion

The work used the probabilistic representation of the solutions of boundary-value problems with discrete (grid) approximations the original stochastic process. The processes are easy for computing, don't require large amounts of memory of computer, but at the same time are comparatively complex because of the need to simulate "long" random trajectories.

It is known, for solving nonlinear boundary problems by simulating branching processes, there are currently two practical Monte Carlo schemes which are thoroughly developed.

The first is random walks on the grids with branching and the second is random walks on spheres with branching. When we using the random walks on the grid with branching or random walks on spheres with branching, we note that when solving problem at an isolated point, we do not have to determine the solution at all the points on the grid.

This is one of fundamental advantages of using the Monte Carlo method to solve nonlinear boundary value problems. Sometimes in constructing algorithms mostly used mean value theorems for parabolic equations and applied the analogy of algorithms like "random walk on spheres" and "random walk on spheroids" with branching. Obtained results can be used for further studying and constructing Monte Carlo algorithms for the solution more general diffusion BVP with polynomial nonlinearities. In future we want to apply these results for the solution to the most complicated practical engineering problems.

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